UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

BOUNDARY VALUE PROBLEMS FOR SEMILINEAR EVOLUTION EQUATIONS OF COMPACT TYPE.

Herbert C. Sager

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The fear of the Lord is the beginning of knowledge. Pr.1:7.

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CONTENTS

BOUNDARY VALUE PROBLEMS FOR SEMILINEAR EVOLUTION EQUATIONS OF COMPACT TYPE

SYNOPSIS

We consider the following evolution equations in a Hilbert space with boundary conditions as stated.

(i)
$$x' + (A(t) + B(t,x))x = f(t,x)\ , \quad t \in J = [0,T] \tag{1}$$
$$x(0) = x(T) \tag{2}$$

(ii)
$$x' + (A(t) + B(t,\zeta(x)))x = f(t,x) \qquad t \in J \tag{3}$$
$$kx(0) = x(T)\ , \qquad k \in \mathbb{R} \tag{4}$$

(iii)
$$x'' = (A + B(t,x,x'))x - f(t,x,x') \qquad t \in J \tag{5}$$
$$\alpha_{i1}x(0) + \alpha_{i2}x'(0) + \beta_{i1}x(T) + \beta_{i2}x'(T) = 0 \quad (i = 1,2) \tag{6}$$

It will be supposed that the linear operators $A(t)$ in the equations (1) and (3) generate analytic and compact semigroups and that the linear operator A in (5) is such that $A^{1/2}$ generates an analytic and compact semigroup. The function f in equations (1), (3) and (5) may have asymptotically sublinear growth. In equations (1) and (5) the perturbation operator B is assumed to be bounded, whereas in equation (3) B may be unbounded but subordinate to $A(t)$. The function ζ in (3) maps $L^2(J,H)$ into $C(J,H)$. It is defined in Chapter II (1.10). We require that k in (4) be sufficiently large. In all three cases we establish the existence of a weak solution under suitable conditions, with the help of Schauder's fixed point theorem.

Many authors have dealt with the Cauchy problem for nonlinear evolution equations in abstract spaces [e.g. 22, 23, 24, 25, 26, 39, 52, 60, 63, 64, 66, 74] or with boundary value problems for linear evolution equations in abstract spaces [e.g. 10, 18, 21, 28, 29, 43, 46].

In the finite dimensional case boundary value problems (linear and nonlinear) for nonlinear differential equations have been studied intensively. See, for example [2, 13, 33, 34, 44, 47, 57, 58, 62, 70, 77, 79].

This type of problem has been generalized to evolution equations in Banach and Hilbert spaces. Amann [1; p454] proves the existence of a periodic solution of $x' + A(t)x = f(t,x)$, using hypotheses which are quite different from ours. Ward [76, 78] discusses quite general linear and nonlinear boundary conditions for the equation $x' = A(t)x + f(t,x)$, but he treats only the asymptotically sublinear case. Kartsatos [35] obtains solutions on $[0,\infty)$ for the abstract boundary value problem $x' + A(t)x = F(t,x)$ and $Ux = b$, where U is a bounded linear operator and $A(t)$ are closed linear with constant domain. His nonlinear part $F(t,x)$ can also be of linear order, but he requires much stronger regularity conditions than we do and a certain positiveness condition on $A-\mu F$ $(\mu \in [0,1])$. His class of boundary conditions does not include periodicity conditions as we have them. Prüss [65] investigates the problem of the existence of T-periodic solutions of the evolution equation $x' = Ax + f(t,x)$, in closed, bounded and convex subsets D of a Banach space. A generates a C_0-semigroup $U(t)$, and $f(t,x)$ is continuous, T-periodic and satisfies a boundary condition relative to D. Further he requires compactness conditions and the boundedness of $f(t,x)$ or the fact that $U(t)$

leaves D invariant. Bruck [9] obtains existence and uniqueness for the two-point boundary value problem $u'' \in Au + f(t)$, $u(0) = x$, $u(T) = y$, for a maximal, monotone, multivalued operator A. Zecca and Zezza [83] consider the multivalued differential problem $x' - A(t)x \in F(t,x)$ and $Lx = Mx$, where M is a nonlinear, continuous operator, $F(t,x)$ is of linear order in x, and $A(t)$ are bounded and linear (see [83; (c),(d),p348]). Becker [8] obtains results on abstract Hammerstein equations which he applies to the periodic boundary value problem $x'' + A^2x = B(t,x)x + f(t,x)$ and $x(0) = x(T)$, $x'(0) = x'(T)$ to prove the existence of a mild solution. In his case $B(t,x)$ is a uniformly bounded linear operator and $f(t,x)$ is a bounded function. They do not depend on x'. The assumptions on A are essentially the same as in our case in Chapter III, (see Travis and Webb [74; Prop.2.6] and Rankin [67; p378]). In [6] Becker shows, using the same assumptions on $B(t,x)$ and $f(t,x)$ as in [8], that the equation $x' + (A + B(t,x))x = f(t,x)$ has a mild periodic solution. Here A generates a compact C_0-semigroup. These results may also be generalized to functional differential equations of retarded type, see Becker [7]. In these papers (except in [9]) fixed point theorems were employed.

The basic motivation for this work comes from the papers by Opial [62] and Becker [6]. For first order systems of equations in $\mathbb{R}^n$, Opial shows that the boundary value problem for the nonlinear equation

$$x' = A(t,x)x + b(t,x) \quad \text{and} \quad Dx = r \tag{7}$$

has a solution. Here D is a bounded linear map from the space C^n of all continuous vector functions to $\mathbb{R}^n$. He introduces the special norm

$$\|A(t)\|_0 = \max\{\Sigma^n_{i,j=1}\left|\int_0^t a_{ij}(s)ds\right| \; ; \quad t \in [0,T]\}$$

in the linear space $L_1^{n\times n}$, consisting of all real $n\times n$ matrix functions $A(t)$ defined and integrable in $[0,T]$ and assumes

(i) The linearized, homogeneous problem

$$x' = A(t,\psi(t))x \quad \text{and} \quad Dx = 0$$

has only the trivial solution for all $\psi(t)$ in C^n.

(ii) $A(t,\psi(t))$ lies in a closed bounded (and hence compact) subset of $L_1^{n\times n}$, for all $\psi(t)$ in C^n.

(iii) Both $A(t,x)$ and $b(t,x)$ satisfy Carathéodory type conditions, and $\beta_\ell(t) = \sup\{|b(t,x)| \; ; \; |x| \le \ell\}$, $\ell = 1,2,\ldots.$ has asymptotically sublinear growth (in ℓ).

With this he can show that the mapping G, which assigns to each continuous $\psi(t)$ the unique solution to the linearized version of (7), is continuous and compact. Together with (iii) Schauder's fixed point theorem can then be applied, hence guaranteeing that the boundary value problem (7) has a solution.

For further developments of this type, see Kartsatos [34], Anichini [2]. We mention also that Mawhin and Ward [57] have used a similar technique.

With basically the same type of hypotheses Becker [6] extends this type of result to certain semilinear evolution equations in separable Hilbert spaces. He shows that the periodic boundary value problem

$$x' + (A + B(t,x))x = f(t,x) \quad \text{and} \quad x(0) = x(T)$$

has a weak solution. Here A is the generator of a compact semigroup, $B(t,x)$ is a bounded linear operator and $f(t,x)$ a bounded function.

In extending techniques applicable to differential equations in $\mathbb{R}^n$, to equations in Hilbert or Banach spaces, one needs to make stronger continuity hypotheses on the coefficients, to ensure the same existence results. One can assume for example that $f(t,x)$ satisfies a Lipschitz condition, or as Pazy [63] has shown, one may suppose that the operator A generates a compact semigroup. In the latter case continuity is sufficient to guarantee local existence. For extentions of these results we refer to Fitzgibbon [23, 24] and Lightbourne and Martin [48].

We also make this compactness assumption throughout.

In generalizing the method that is implied by assumption (ii) above, Becker considers the set of all strongly measurable operator functions from $[0,T]$ to $\mathcal{B}(H)$, the bounded linear operators in H, with $\|B(t)\| \leq N$, for almost every $t \in [0,T]$. This set however is not compact in the strong operator topology, but it is compact in the weak operator topology on $L^2(J,H)$, see [6; p36].

The compact character of the semigroup implies that certain integral operators are compact in $L^2(J,H)$, see Laptev [45]. This fact ensures strong convergence of solutions in his case, if $B_n(t)$ converges only weakly. Further the weak compactness of the above mentioned set enables him to do without a priori estimates of the derivative which are not easily available in the case of unbounded operators A. Such estimates are required by most authors, see for example [9, 10, 39].

The present work is divided into three chapters. In *Chapter* I we extend Becker's result to variable operators $A(t)$ and further we allow the function $f(t,x)$ to be asymptotically sublinear in all three chapters (see for example (3.3)). We show that the semilinear equation

(1) has a weak periodic solution. In *Chapter* II we consider unbounded perturbations $B(t,x)$ of the type described in Kato and Tanabe [41], and show that for such $B(t,x)$ and certain smooth functions ζ (see (1.10)), equation (3) with quasi-periodic boundary condition (4) has a solution for sufficiently large k. The example we give contains new results concerning a time dependent Sturm-Liouville operator.

In *Chapter* III, Becker's result is extended to second order evolution equations with general boundary conditions as described for example in Krein [43; p251]. We show that equation (5) has a weak solution which satisfies (6).

Throughout, we require Carathéodory type regularity conditions on $f(t,x)$ and $B(t,x)$ (see Krasnoselskii [42; p20]).

Since we need the properties of a Hilbert space and since the inhomogeneous part $f(t,\rho(t))$ belongs to $L^2(J,H)$, we work with weak solutions. In Chapter I the notion of a weak solution according to Kato and Tanabe [41] is used, (see also Lions [49]). As in their situation the unbounded operator $A(t)$, as well as its domain $\mathcal{D}(A(t))$, may vary with t. They assume however (see [41; (E.1)]) that for $t \in [0,T]$, $-A(t)$ generates an analytic semigroup. This rather strong assumption is not needed in Becker's theory. It remains an open question as to whether these assumptions in Chapter I may be replaced by the more general 'hyperbolic' assumptions on $A(t)$, namely that $-A(t)$ generate C_0-semigroups only, according to Kato [37, 38] or Heyn [30]. There the domain may also vary in time. For a treatment of the Cauchy problem for the quasi-linear evolution equation $x' + A(t,x)x = f(t,x)$, using these assumptions, we refer to the well-known exposition by Kato [39] and to an extention of his result by

Murphy [60]. For another approach to hyperbolic evolution equations we refer to Ishii [32]. Another development in the direction of operators $A(t)$ with variable domain, using variable closed sets and a generalized subtangential condition, is due to Prüss [64]. Again however, he requires analyticity.

In Chapter II, where we consider unbounded perturbations of $A(t)$, it is natural to assume that $-A(t)$ generate analytic semigroups, since otherwise $-(A(t)+B(t))$ may not even be the generator of a C_0-semigroup, even for $B(t)$ subordinate to $A(t)$, Kato [36; p497-507]. The operator A in Chapter III need not generate an analytic semigroup, but our assumptions imply that $A^{1/2}$ generates such a semigroup (see Krein [43; p250], Balakrishnan [4]). The notion of a weak solution used in this chapter is derived from the corresponding one for first order evolution equations (see Balakrishnan [3; p204], Ball [5]). For other notions of a weak solution we refer to Zaidman [82; p59], Raskin and Sobolevskii [66; p62].

In all three chapters we work in separable Hilbert spaces. This is inevitable in Chapters I and III, since we require Proposition 1.1 of Becker [6; p36]. It appears that Chapter II could be treated in a reflexive Banach space setting. If this were possible, the rather restrictive assumption on γ (see (AII.2)) could be relaxed.

In Chapters I and III we suppose as in Opial's assumption (i) above, that the linearized version of (1), (2) (and (5), (6)) has a unique solution for all $B(t,\psi(t))$, $\psi(t) \in L^2(J,H)$. Various conditions are known which imply this uniqueness. We refer here to the well known criterion given by Lazer and Leach [47] and Landesman and Lazer [44]. In [44] it is shown that the boundary value problem for the nonlinear elliptic partial differential equation

$$\Delta u + p(x,Du)u = h(x,Du) \tag{8}$$

$$u(x) = g(x) \quad \text{on} \quad \partial\Omega$$

has a solution, for a given continuous g. Here $\Omega \subset \mathbb{R}^n$ is a Dirichlet domain, Δ is the n-dim. Laplacian and Du denotes the vector consisting of u and all its first derivatives. They assume that p and h are continuous and bounded functions and that for some integer $N \geq 1$, there are numbers γ_N and γ_{N+1} such that

$$\lambda_N < \gamma_N \leq p(x,Du) \leq \gamma_{N+1} < \lambda_{N+1} \quad \text{for all} \quad (x,Du), \tag{9}$$

where $\lambda_1 \leq \lambda_2 \leq \ldots.$ denote the eigenvalues of the linear homogeneous problem

$$\Delta u + \lambda u = 0 \,, \quad u = 0 \quad \text{on} \quad \partial\Omega \,.$$

The nonresonance condition (9) implies that the linearized version of (8) has a unique solution for any $h \in L^2(\Omega)$ and that a certain estimate holds independently of p.

This idea has received much attention, and we refer for example to Kannan and Locker [33] and Ward [77, 80] for further developments. The newest development in this direction is due to Mawhin and Ward [57]. In their paper they weaken condition (9) but retain nonresonance. The nonlinearity p does no longer have to be uniformly bounded away from the eigenvalues. It is required instead that $p(x,Du)$ be asymptotically between two eigenvalues, i.e. for some integer N and functions $\alpha, \beta \in L^\infty(\Omega)$ they assume that

$$\lambda_N \leq \alpha(x) \leq \underline{\lim}_{|u|\to\infty} p(x,Du) \leq \overline{\lim}_{|u|\to\infty} p(x,Du) \leq \beta(x) \leq \lambda_{N+1}$$

uniformly for $x \in \Omega$ and the non-u components of Du. The proof uses Leary-Schauder degree theory. Such methods have gained considerable importance lately, we refer to Mawhin [53, 54, 55, 56] for this aspect.

In the abstract case we mention Bruck [9]. From his results it follows that for the special case where A is linear and accretive, there can be only one solution to the boundary value problem

$$x'' = Ax + f(t) \quad \text{a.e. } t \in (0,T) \quad \text{and} \quad x(0) = x \, , \; x(T) = y \, ,$$

x and y belonging to a real Hilbert space. In Dunninger and Levine [18] the equation $x'' + Nx = 0$ $(t \in [0,T])$ is shown to have only the zero solution which satisfies the boundary conditions

$$\begin{aligned} -\cos\theta_1 \, x'(0) + \sin\theta_1 \, x(0) &= 0 \\ \cos\theta_2 \, x'(T) + \sin\theta_2 \, x(T) &= 0 \end{aligned} \tag{10}$$

if and only if λ_ℓ is not an eigenvalue of N (N is closed, linear with dense domain in a Banach space). Here λ_ℓ denote the eigenvalues of the scalar eigenvalue problem $\psi'' + \lambda\psi = 0$, $\psi(t)$ satisfying (10). Another result in this direction is given by Burak [10]. There uniqueness of solution of a two-point boundary value problem is achieved by choosing the interval $[0,T]$ sufficiently small. In the examples we use a variant of Landesman and Lazer's criterion, as Becker [6; p35,45] has already done.

In Chapters I and III we obtain the necessary convergence results by the use of the compactness of the evolution operators and associated integral operators, and the weak compactness property of a Hilbert space. Even though the method of proof of these results (Theorem 2.6 in Chapter I and Theorem 3.1 in Chapter III) is similar, there are however considerable differences in the spaces that are required in their proofs and in the use of the uniqueness and the boundary conditions. The results, especially the ones in Chapter I, are similar to the corresponding ones in Becker [6; p39]. The type of convergence result in Chapter II (Theorem 1.7) is different. It involves the nonlinearity already and requires regularity conditions on the basic assumption (AII.2).

Further in Chapters I and III the compactness of the set of all $B(t) \in \mathcal{B}(H)$ with $\|B(t)\| \leq N$ (a.e. for $t \in [0,T]$) in the weak topology in $\mathcal{B}(L^2(J,H))$, (see [6; p36]), is instrumental in establishing uniform bounds for certain operators associated with the boundary conditions (2) and (6). In this way we do not need a priori estimates of the derivative which are needed by many authors.

In Chapter III, our approach is motivated by Krein [43; pp249-270], i.e. second order equations are equivalent to first order systems (see also Fitzgibbon [25]). Another well known approach to second order evolution equations is by means of cosine families. These were introduced by Segal [69] and developed by Fattorini [19, 20]. We refer to Travis and Webb [74] who discuss also the connection between the cosine family and the first order systems approach to second order equations. Finally we mention the developments of Krein's results [43; pp249-270] by Gershtein and Sobolevskii [28] and by Heimes [29].

The general notation that is used throughout is introduced in Section 1 of the first chapter. In referring for instance to Theorem 1.3 (or eqn. (2.1)) in Chapter I, we would write I Theorem 1.3 (or I(2.1)), if this reference is not made in Chapter I.

CHAPTER I

0. INTRODUCTION

This chapter is concerned with the periodic boundary value problem for the semi-linear evolution equation

$$\begin{aligned} x' + (A(t) + B(t,x))x &= f(t,x) \\ x(0) &= x(T) \end{aligned} \tag{0.1}$$

in a separable Hilbert space H. $A(t)$ is a function from $J = [0,T]$ to the set of unbounded operators acting in H. We are working with weak solutions according to Kato and Tanabe [41]. Hence $A(t)$ needs to satisfy their assumptions (E.1), (E.2), (E.3) only. However we suppose further that $A(t)$ generates a compact semigroup for every $t \in J$. The perturbation $B(t,x)$ is assumed to be a bounded linear operator in H, as in Becker [6], and $f(t,x)$ a function with values in H and having sublinear growth. We will prove the existence of a weak periodic solution (Theorem 3.1). The chapter is divided into four sections. Section 1 contains four important preliminary results, three of which are well known. In Section 2 an evolution operator for the perturbed equation $x' + (A(t) + B(t))x = 0$ will be constructed and convergence properties of this equation obtained. Some of these depend on the compactness of certain integral operators and are similar to Becker [6; p39].

The uniqueness assumption of Theorem 3.1 together with the compactness of the evolution operator guarantee a unique weak periodic solution to $x' + (A(t) + B(t))x = h(t)$, $h(t) \in L^2(J,H)$. The linear operator associated with this periodic boundary value problem depends on $B(t)$. One can show that this operator is uniformly bounded for $B(t)$ in S , S being the set of bounded, strongly measurable operator functions $B(t)$ such that $\|B(t)\| \leq N$ (a.e. for $t \in J$) , which is compact in the weak operator topology in $L^2(J,H)$, (Becker [6; p36,42]).

These results guarantee a unique weak periodic solution to the linearized version of (0.1). Schauder's fixed point theorem can then be applied to prove the existence of a weak periodic solution of (0.1). This forms the content of Section 3. Finally in Section 4 we will illustrate the result in the previous section. $A(t)$ will represent a strongly elliptic partial differential operator, which need not be self-adjoint. The uniqueness assumption of Theorem 3.1 will be satisfied provided the norm of $[B(t,\rho(t)) - \ell I]$, $(\rho(t) \in L^2(J,H))$ is sufficiently small.

We mention here that the continuity in the uniform operator topology of the evolution operators Φ of (2.4) and Ψ of (2.1), (see Theorems 2.2 and 2.4) is not needed in the proof of the main result. We have included the results because they are of interest in themselves.

1. NOTATION AND PRELIMINARY RESULTS

We work in Hilbert space, and all Hilbert spaces will be supposed separable. H, H_1, H' etc. will be used to denote Hilbert spaces.

Given Hilbert spaces H_1 and H_2, $\mathcal{B}(H_1,H_2)$ will denote the bounded linear operators from H_1 to H_2, and $\mathcal{B}(H)$ will be written instead of $\mathcal{B}(H,H)$.

For $J = [0,T] \subseteq \mathbb{R}$ (the reals) a compact interval, we denote by $L^p(J,H)$ $(1 \le p < \infty)$ the space of (equivalence classes of) functions from J to H which are strongly measurable and satisfy $\int_J \|x(t)\|^p dt < \infty$. The inner product in H will in these circumstances be denoted by $(.,.)$ and that in $L^2(J,H)$ by $[.,.]$ where $[x(t),y(t)] = \int_J (x(t),y(t))dt$. With this inner product, $L^2(J,H)$ becomes a separable Hilbert space. We write $\|x(t)\|_{L^2} = [x(t),x(t)]^{1/2}$.

Let $\mathcal{L}(J,H)$ denote the set of maps from $J \subseteq \mathbb{R}$ to $\mathcal{B}(H,H)$ and $M(J,H)$ the subset of strongly measurable such maps (i.e. those $B(t) : J \to \mathcal{B}(H,H)$ such that $B(t)x$ is strongly measurable for each $x \in H$). $\|B(t)\|$ denotes the Hilbert space H-norm of the operator $B(t)$. For details of the measure theory, see Dunford and Schwartz [17;Vol.I].

If $B(t) \in M(J,H)$ and $\|B(t)\| \le N (t \in J)$ then for almost all t, $B(t) : H \to H$ and in addition we can regard $B(t) : L^2(J,H) \to L^2(J,H)$ as defined by $B(t)(x(t)) = B(t)x(t)$ (a.e. for $t \in J$). We denote the latter (bounded linear) transformation by $\tilde{B}$.

The following proposition is due to Becker [6; p36]. We will need it in the proof of the main theorem (Theorem 3.1).

Proposition 1.1. *For any* $N > 0$, *the set*

$S_N = \{\tilde{B} \mid B(t) \in M(J,H)$ and $\|B(t)\| \leq N$ for a.e. $t \in J\}$

is compact in the weak topology in $\mathcal{B}(L^2(J,H))$.

The next result which is due to Laptev [45;p596], will be used throughout. Here it is quoted in the form given by Becker [6 ;p36,38].

Proposition 1.2.

(a) *Let* $K(t,s) : J \times J \to \mathcal{B}(H)$ *be compact for almost all* (t,s), *and let* $\int_{J\times J} \|K(t,s)\|^2 ds dt < \infty$. *Then the map in* $L^2(J,H)$ *defined by*

$$x(t) \quad \to \int_J K(t,\tau)x(\tau)d\tau$$

is compact.

(b) *Let* $K(t,s) : J \times J \to \mathcal{B}(H)$ *be compact for almost all* (t,s), *and let* $\int_J \|K(t,s)\|^2 ds < \infty$ $(t$ *fixed*). *Then the map* $A : L^2(J,H) \to H$ *defined by* $Ax = \int_J K(t,\tau)x(\tau)d\tau$ *is compact for each* t *for which it is defined.*

We state here another result, due to Krasnoselskii [42;p20-26]. The proof given there for $\mathbb{R}$ is equally valid for H. We will need the result in the proof of Theorem 3.1.

Proposition 1.3. *Assume that the function* $f(t,x) : J \times H \to H$ *is continuous in* $x \in H$, *for almost all* $t \in J$, *and measurable with respect to* t *for all values of* $x \in H$. *Assume further that for every* $x(t) \in L^2(J,H)$, $f(t,x(t)) \in L^2(J,H)$. *The operator in* $L^2(J,H)$ *defined by* $x(t) \to f(t,x(t))$ *is then continuous and transforms bounded sets into bounded sets.*

Finally we will prove a proposition generalizing a method which we will be using repeatedly.

Proposition 1.4. *Let* L *and* M *be compact linear operators in a Hilbert space* H . *Further let* (C_n) *be a bounded sequence in* $B(H)$. *For given bounded sequences* (x_n) *and* (f_n) *in* H *suppose that there exists a bounded sequence* (y_n) *in* H *such that the equation*

$$y_n = Mx_n - L(C_n y_n - f_n)$$

holds for every positive integer n . *The sequence* (y_n) *then contains a convergent subsequence.*

Proof. Since (y_n) is bounded, $(C_n y_n)$ is bounded. The weak compactness property of H guarantees that there exists a subsequence (j) such that $(C_j y_j - f_j)$ and (x_j) are weakly convergent. The compactness of L and M then implies that (y_j) is convergent $\square$.

Remark. In applying the proposition, the Hilbert space will usually be $L^2(J,H)$ or $L^2([s,T],H)$; L will be identified with the operator $\int_s^t \Phi(t,\tau) \cdot d\tau$, ($s \in [0,T)$ and fixed), which is compact if $\Phi(t,s)$ is as in Proposition 2.5; $C_n = \tilde{B}_n$ which is bounded by N and the operator M will be defined by

$$x(t) \to (T-s)^{-1}\Phi(t,s)\int_s^T x(\tau)d\tau .$$

Since $\Phi(t,s)$, as given in Proposition 2.5, is uniformly bounded and compact for $s < t$, it follows easily that M maps bounded sets into relatively compact sets in $L^2([s,T],H)$. Hence M is compact.

2. PERTURBED EQUATIONS OF EVOLUTION

This section will be concerned with the abstract evolution equation

$$x' + (A(t) + B(t))x = 0, \qquad (' = d/dt) \tag{2.1}$$

in a separable Hilbert space H. A(t) is a function from $J = [0,T]$ to the set of unbounded operators acting in H. It is assumed to satisfy the conditions (E.1), (E.2), (E.3) of Kato, Tanabe [41;p109]. Thus the domain of A(t) may vary with t. Furthermore A(t) generates a compact semigroup for every $t \in J$.

B(t) is a strongly measurable map from J to the set of bounded operators $\mathcal{B}(H)$ in H and $\|B(t)\| \leq N$ for almost every t in J.

Following [41;p111] (see also Lions [49]) we shall introduce the notion of a weak solution and show that the equation $x' + A(t)x = 0$ is of compact type (see Definition 2.1). It will then be shown, using methods similar to Balakrishnan [3;p221] that equation (2.1) is also of compact type (Theorem 2.3).

The remaining part of the section will be concerned with convergence properties of equation (2.1). These are similar to the ones found in Becker [6;p39].

We begin by stating all our assumptions. The first three can be found in [41;p109,110] and also in Tanabe [72;p117,129]. In what follows we shall denote by Σ a fixed, closed angular domain.

$$\Sigma = \{\lambda \in \mathbb{C};\ |\arg(\lambda)| \le \pi/2 + \theta,\ 0 < \theta < \pi/2\}$$

(AI.1) For each $t \in J = [0,T]$, $A(t)$ is a closed, linear operator, defined densely in a separable Hilbert space H. The resolvent set $\rho(-A(t))$ of $-A(t)$ contains Σ, and the resolvent satisfies

$$\|(\lambda I + A(t))^{-1}\| \le M'/|\lambda|$$

for any $\lambda \in \Sigma$ and $t \in J$, where M' is a positive constant independent of λ and t.

(AI.2) $A(t)^{-1}$, which is a bounded operator for each t, is continuously differentiable with respect to $t \in J$ in the uniform operator topology.

(AI.3) There exist two constants $N' > 0$ and $0 \le \rho < 1$ such that the inequality

$$\|(\partial/\partial t)(\lambda I + A(t))^{-1}\| \le N'/|\lambda|^{1-\rho}$$

holds for every $\lambda \in \Sigma$ and $t \in J$.

(AI.4) For every $t \in J$ there is some λ in $\rho(-A(t))$ such that $(\lambda I + A(t))^{-1}$ is a compact linear operator.

Definition 2.1. The equation $x' + A(t)x = 0$ is of *compact type*, if there exists an evolution operator $\Phi(t,s)$ with the following properties.

(C.1) $\Phi(t,s)$ is a strongly continuous map from $S = \{(t,s) \in J^2 | 0 \le s \le t \le T\}$ into $\mathcal{B}(H)$;
$\|\Phi(t,s)\| \le M$ on S;

$\Phi(t,t) = I$; $\Phi(t,\sigma)\Phi(\sigma,s) = \Phi(t,s)$ $\quad (0 \le s \le \sigma \le t \le T)$

(C.2) For every $f \in L^2(J,H)$, $s \in [0,T)$, and $u(s) \in H$, there exists an unique continuous function $u(t)$ in $[s,T]$ given by

$$u(t) = \Phi(t,s)u(s) + \int_s^t \Phi(t,\sigma)f(\sigma)d\sigma \tag{2.2}$$

such that

$$\int_s^T (u(t),\eta'(t) - A^*(t)\eta(t))dt + \int_s^T (f(t),\eta(t))dt + (u(s),\eta(s)) = 0 \tag{2.3}$$

where $\eta(t)$ is any function satisfying

(i) for each $t \in J$, $\eta(t)$ belongs to $\mathcal{D}(A^*(t))$.

(ii) $\eta(t)$, $\eta'(t)$ and $A^*(t)\eta(t)$ are strongly continuous in $[s,T]$.

(iii) $\eta(T) = 0$. (For the existence of such functions η we refer to [72;p132]).

A function $u(t)$, continuous in $[s,T]$ and satisfying eqn. (2.3) for any test function $\eta(t)$ is called a *weak solution* of $x' + A(t)x = f(t)$ in $(s,T]$. (c.f.[41;p111])

(C.3) For $t > s$, $\Phi(t,s)$ is a compact, linear operator.

Theorem 2.1 *If* $A(t)$ *satisfies the assumptions* (AI.1) - (AI.4), *then* $x' + A(t)x = 0$ *is of compact type.*

Remark. (AI.1) implies that for each $t \in J$, $-A(t)$ generates an analytic semigroup $\exp(-sA(t))$. With assumptions (AI.1) to (AI.3) an evolution operator $\Phi(t,s)$ can then be constructed:

$$\Phi(t,s) = \exp(-(t-s)A(t)) + \int_s^t \exp(-(t-\sigma)A(t))R(\sigma,s)d\sigma \tag{2.4}$$

where $R(t,s)$ is determined as the solution of the integral equation

$$R(t,s) - \int_s^t R_1(t,\sigma)R(\sigma,s)d\sigma = R_1(t,s).$$

Here we have put $R_1(t,s) = -(\partial/\partial t + \partial/\partial s)\exp(-(t-s)A(t))$.

It follows that both $R_1(t,s)$ and $R(t,s)$ are continuous for $0 \le s < t \le T$, in the uniform operator topology, and satisfy

$$\|R_1(t,s)\| \le C|t-s|^{-\rho}$$

$$\|R(t,s)\| \le C|t-s|^{-\rho} \tag{2.5}$$

Furthermore it follows that $\Phi(t,s)$ satisfies (C.1) of Definition 2.1. If $f(t)$ is continuous, then (C.2) is also guaranteed. These results are proved in [41;p112-117].

Proof of Theorem 2.1. By the above remark (C.1) is secured. As to (C.2), it remains to show that the result also holds for $f(t)$ in $L^2(J,H)$. The uniqueness follows as in [41;p115]. Thus we only need to verify that for $f(t) \in L^2(J,H)$ and any function $\eta(t)$ satisfying (i), (ii), (iii) of (C.2), we refer to [41;(3.14)],

$$\int_s^T\left(\int_s^t \Phi(t,\sigma)f(\sigma)d\sigma,\ \eta'(t) - A^*(t)\eta(t)\right)dt + \int_s^T (f(\sigma),\eta(\sigma))d\sigma = 0.$$

Let (f_n) be a sequence of continuous functions in $C(J,H)$ with $f_n \to f$ in $L^2(J,H)$. By [41;(3.14)] we have for any n

$$H_\eta(f_n) = \int_s^T\left(\int_s^t \Phi(t,\sigma)f_n(\sigma)d\sigma,\ \eta'(t) - A^*(t)\eta(t)\right)dt + \int_s^T (f_n(\sigma),\ \eta(\sigma))d\sigma = 0$$

Thus $|H_\eta(f)| \le |H_\eta(f) - H_\eta(f_n)| + |H_\eta(f_n)| = |H_\eta(f) - H_\eta(f_n)|$,

and

$$\begin{aligned} |H_\eta(f) - H_\eta(f_n)| &\le \left|\int_s^T\int_s^t (\Phi(t,\sigma)[f(\sigma) - f_n(\sigma)],\eta'(t) - A^*(t)\eta(t))d\sigma dt\right| \\ &\quad + \left|\int_s^T ([f(\sigma) - f_n(\sigma)],\eta(\sigma))d\sigma\right| \\ &\le M\cdot C_\eta T\int_0^T \|f_n(\sigma) - f(\sigma)\|d\sigma \to 0 \quad (n \to \infty). \end{aligned}$$

The function $u(t)$ given by (2.2) is continuous in $[s,T]$. This can be seen as follows. Let $t_2 < t_1$, then

$$u(t_1) - u(t_2) = [\Phi(t_1,s) - \Phi(t_2,s)]u(s) +$$

$$\int_s^{t_2} [\Phi(t_1,\sigma) - \Phi(t_2,\sigma)]f(\sigma)d\sigma + \int_{t_2}^{t_1} \Phi(t_1,\sigma)f(\sigma)d\sigma.$$

The first and the last terms on the right hand side of the above equation clearly tend to zero in norm as $|t_1 - t_2| \to 0$.

If t_2 is fixed, $t_2 < t_1$, then because of (C.1)

$$\int_s^{t_2} \| [\Phi(t_1,\sigma) - \Phi(t_2,\sigma)]f(\sigma)\| d\sigma \to 0 \quad \text{as } t_1 \to t_2,$$

using the dominated convergence theorem. If t_1 is fixed, again with $t_2 < t_1$, we define the function

$$F(t_2,\sigma) = \| [\Phi(t_1,\sigma) - \Phi(t_2,\sigma)]f(\sigma)\| \quad \text{for } s \le \sigma \le t_2$$

$$= 0 \quad \text{for} \quad t_2 < \sigma \le t_1$$

so that

$$\int_s^{t_2} \| [\Phi(t_1,\sigma) - \Phi(t_2,\sigma)]f(\sigma)\| d\sigma = \int_s^{t_1} F(t_2,\sigma)d\sigma.$$

The function $F(t_2,\sigma)$ is bounded by $2M\|f(\sigma)\|$ which is integrable. In order to apply the dominated convergence theorem, it remains to show that for almost every $\sigma \in (s,t_1)$

$$\lim_{t_2 \to t_1} F(t_2,\sigma) = 0 .$$

Since σ remains fixed while t_2 approaches t_1, we may assume that $\sigma < t_2 < t_1$ for almost every $\sigma \in (s,t_1)$. Therefore

$$F(t_2,\sigma) = \| [\Phi(t_1,\sigma) - \Phi(t_2,\sigma)]f(\sigma)\|$$

which does tend to zero as $t_2 \to t_1$. Thus (C.2) is verified.

Next it is shown that under the assumption (AI.4), the statement in (C.3) is correct. The analytic semigroup $\exp(-\tau A(t))$ generated by $-A(t)$ is differentiable with respect to τ, for $\tau > 0$, in $\mathcal{B}(H)$, we refer to [72;p130]. Hence it follows that for $\tau > 0$, $\exp(-\tau A(t))$ is continuous in the uniform operator topology. By (AI.4) and a theorem by Pazy [63;p24], the semigroup $\exp(-\tau A(t))$ is compact for $\tau > 0$. Accordingly, $\exp(-(t-\sigma)A(t))$ is a compact linear operator for any $\sigma < t$. Furthermore for $s < \sigma < t$, $R(\sigma,s)$ is bounded and linear, so that $\exp(-(t-\sigma)A(t))R(\sigma,s)$ is compact. Let (x_n) denote a sequence in H, converging weakly to x_0. It follows that

$$\exp(-(t-\sigma)A(t))R(\sigma,s)x_n \to \exp(-(t-\sigma)A(t))R(\sigma,s)x_0$$

strongly as $n \to \infty$, for almost all $\sigma \in [s,t]$. Moreover by (2.5), and since (x_n) is a bounded sequence

$$\| \exp(-(t-\sigma)A(t))R(\sigma,s)x_n\| \leq C|\sigma - s|^{-\rho}$$

which is integrable. Thus, by the dominated convergence theorem

$$\int_s^t \exp(-(t-\sigma)A(t))R(\sigma,s)x_n d\sigma \to \int_s^t \exp(-(t-\sigma)A(t))R(\sigma,s)x_0 d\sigma$$

strongly as $n \to \infty$, and by (2.4), $\Phi(t,s)x_n \to \Phi(t,s)x_0$ $(n \to \infty)$, i.e. for $s < t$, $\Phi(t,s)$ is compact. For a different proof of the compactness see [23, 24]. This completes the proof of Theorem 2.1 □ .

We now prove another property about $\Phi(t,s)$, namely

Theorem 2.2*. $\Phi(t,s)$ *is continuous on* $S' = \{(t,s) \in J^2 | 0 \leq s < t \leq T\}$ *in the uniform operator topology.*

Proof. It requires the continuity of $T(t,s) = \exp(-(t-s)A(t))$ and

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*As remarked earlier, this result is not needed for the proof of the main theorem.

$R(t,s)$ in (2.4) in the uniform operator topology. By [41;(2.6),(2.7)] the operator $T(t,s)$ is differentiable with respect to s and t in S'. Therefore it is continuous there in the uniform operator topology. Set $\|T(t,s)\| \leq D$ for $0 \leq s \leq t \leq T$.

It remains to show that the integral in the expression of (2.4) for $\Phi(t,s)$ has the desired continuity property. This is now verified separately for t and s.

(i) Continuity in t

Let $s < t_2 \leq t_1$ and let t_2 be fixed. Then

$$I_0 = \left\| \int_s^{t_1} T(t_1,\sigma)R(\sigma,s)d\sigma - \int_s^{t_2} T(t_2,\sigma)R(\sigma,s)d\sigma \right\|$$

$$\leq \int_s^{t_2} \|T(t_1,\sigma) - T(t_2,\sigma)\| \, \|R(\sigma,s)\| d\sigma + \int_{t_2}^{t_1} \|T(t_1,\sigma)R(\sigma,s)\| d\sigma.$$

Denote the integrals on the right hand side of the above inequality by I_1 and I_2 respectively. The integrand in I_2 is integrable and thus I_2 tends to zero as $t_1 \to t_2$. The integrand in I_1 converges to zero as $t_1 \to t_2$ for almost all $\sigma \in (s,t_2)$. Furthermore the integrand is bounded by $2DC(\sigma-s)^{-\rho}$, which is integrable. From the dominated convergence theorem it follows that $I_0 \to 0$ $(t_1 \to t_2)$.

Now let t_1 be fixed in $s < t_2 \leq t_1$. Thus $t_1 - s$ is a given positive number. We may assume that t_2 is such that $t_1 - t_2 < (t_1-s)/3$. To a given ε, $0 < \varepsilon < (t_1-s)/3$ we choose κ with $0 < \kappa < \min\{\varepsilon, (\varepsilon(1-\rho)/6CD)^{1/1-\rho}\}$. Thus $T(t,s)$ is uniformly continuous in $\{(t,s) \in J^2 \mid 0 \leq s \leq t - \kappa \text{ and } \kappa \leq t \leq T\}$, i.e. we can find δ, $0 < \delta < (\varepsilon(1-\rho)/3CD)^{1/1-\rho}$ such that $|t_1 - t_2| < \delta$ implies $\|T(t_1,\sigma) - T(t_2,\sigma)\| < \varepsilon(1-\rho)/3C(t_1-s)^{1-\rho}$ for any $\sigma \in [s,t_2 - \kappa]$. Then

$$I_0 \le \int_{t_2}^{t_1} \|T(t_1,\sigma)R(\sigma,s)\| d\sigma + \left(\int_s^{t_2-\kappa} + \int_{t_2-\kappa}^{t_2}\right) \|T(t_1,\sigma) - T(t_2,\sigma)\| \|R(\sigma,s)\| d\sigma$$

$$\equiv I_1 + I_2 + I_3$$

$$I_1 \le CD\int_{t_2}^{t_1} (\sigma-s)^{-\rho} d\sigma = (CD/1-\rho)\{(t_1-s)^{1-\rho} - (t_2-s)^{1-\rho}\}$$

Since the function $(x+\Delta)^{1-\rho} - x^{1-\rho}$ is decreasing with x,

$$I_1 \le CD(t_1-t_2)^{1-\rho}/1-\rho < \delta^{1-\rho}CD/1-\rho < \frac{\varepsilon}{3}$$

$$I_2 < \varepsilon(1-\rho)/3C(t_1-s)^{1-\rho}\int_s^{t_1} C(\sigma-s)^{-\rho} d\sigma \le \frac{\varepsilon}{3}$$

$$I_3 \le 2CD\int_{t_2-\kappa}^{t_2} (\sigma-s)^{-\rho} d\sigma = (2CD/1-\rho)\{(t_2-s)^{1-\rho} - (t_2-s-\kappa)^{1-\rho}\}$$

Again since the function $x^{1-\rho} - (x-\kappa)^{1-\rho}$ is decreasing with x

$I_3 \le \kappa^{1-\rho}2CD/1-\rho < \varepsilon/3$. Thus $I_0 < \varepsilon$. This proves the continuity of $\Phi(t,s)$ in t. For a different proof, using compactness, we refer to Ward [78].

(ii) Continuity in s.

Again let $s_1 \le s_2 < t$ with s_2 fixed. Then

$$II_0 = \left\|\int_{s_1}^{t} T(t,\sigma)R(\sigma,s_1)d\sigma - \int_{s_2}^{t} T(t,\sigma)R(\sigma,s_2)d\sigma\right\| \le \int_{s_1}^{s_2} \|T(t,\sigma)R(\sigma,s_1)\| d\sigma$$

$$+ \int_{s_2}^{t} \|T(t,\sigma)\| \|R(\sigma,s_1) - R(\sigma,s_2)\| d\sigma \equiv II_1 + II_2 .$$

The integrand of II_1 is integrable and hence II_1 will tend to zero as $s_1 \to s_2$. The integrand in II_2 will converge to zero as $s_1 \to s_2$ for almost all $\sigma \in (s_2,t)$. Further the integrand is bounded by $CD\{(\sigma-s_1)^{-\rho} + (\sigma-s_2)^{-\rho}\} \le 2CD(\sigma-s_2)^{-\rho}$ since $s_1 \le s_2$, and the last expression is integrable. As before we conclude that

$II_0 \to 0 \ (s_1 \to s_2)$.

Now let s_1 be fixed in $s_1 \le s_2 < t$. The number $t - s_1$ is given and fixed. We assume that $s_2 - s_1 < (t-s_1)/3$. To any given ε, $0 < \varepsilon < (t-s_1)/3$ we choose $0 < \kappa < \varepsilon$ such that $\kappa^{1-\rho} < \varepsilon(1-\rho)/6CD$. Because of the uniform continuity of $R(t,s)$ on $S_1 = \{(t,s) \in J^2 \mid s+\kappa \le t \le T \text{ and } 0 \le s \le T-\kappa\}$, we can find δ with $0 < \delta^{1-\rho} \le \varepsilon(1-\rho)/3CD$ such that $\|R(\sigma,s_1) - R(\sigma,s_2)\| < \varepsilon/3DT$, for any $\sigma \in [s_2+\kappa,t]$ as soon as $|s_2 - s_1| < \delta$. Hence

$$III_0 \le \int_{s_1}^{s_2} \|T(t,\sigma)R(\sigma,s_1)\|\,d\sigma + \Big(\int_{s_2}^{s_2+\kappa} + \int_{s_2+\kappa}^{t}\Big)\|T(t,\sigma)[R(\sigma,s_1) - R(\sigma,s_2)]\|\,d\sigma$$

$$\equiv III_1 + III_2 + III_3.$$

$$III_1 \le CD\int_{s_1}^{s_2} (\sigma-s_1)^{-\rho}d\sigma = (s_2-s_1)^{1-\rho}CD/1-\rho < \varepsilon/3$$

$$III_2 \le CD\int_{s_2}^{s_2+\kappa} (\sigma-s_1)^{-\rho} + (\sigma-s_2)^{-\rho}d\sigma = \frac{CD}{1-\rho}\{(s_2-s_1+\kappa)^{1-\rho} - (s_2-s_1)^{1-\rho} + \kappa^{1-\rho}\}$$

$$\le \kappa^{1-\rho}2CD(1-\rho)^{-1} < \varepsilon/3,$$

since the function $(x+\kappa)^{1-\rho} - x^{1-\rho}$ is decreasing in x.

$III_3 < \varepsilon/3$. Thus $III_0 < \varepsilon$. This completes the proof □

Remark. The theorem is true without assuming (AI.4).

The following perturbation result will be needed in Theorem 3.1. We note that the evolution operator for eqn. (2.1) is bounded independently of $B(t)$ for $B(t)$ in S_N of Proposition 1.1.

Theorem 2.3. *Let* $x' + A(t)x = 0$ *be of compact type with evolution operator* $\Phi(t,s)$. *Let* $B(t) \in M(J,H)$ *and satisfy* $\|B(t)\| \le N$ a.e. *for* $t \in J$. *Then*

$$x' + (A(t) + B(t))x = 0$$

is of compact type and its evolution operator satisfies

$$\Psi(t,s)x(s) = \Phi(t,s)x(s) - \int_s^t \Phi(t,\tau)B(\tau)\Psi(\tau,s)x(s)d\tau \tag{2.6}$$

for $s < t$ *and* $x(s) \in H$. *Also* $\|\Psi(t,s)\| \le K$ *where* K *depends only on* M *of* (C.1) *and* N *and the length of the interval* $J = [0,T]$.

The unique weak solution of $x' + (A(t) + B(t))x = f(t)$ *is given by*

$$x(t) = \Psi(t,s)x(s) + \int_s^t \Psi(t,\sigma)f(\sigma)d\sigma \tag{2.7}$$

Remark. For any $x(t) \in L^2(J,H)$, $B(t)x(t) \in L^2(J,H)$. Thus by (C.2) we would expect that if a weak solution $x(t)$ of (2.1) exists, it would satisfy

$$\int_s^T (x(t),\eta'(t) - A^*(t)\eta(t))dt + \int_s^T (-B(t)x(t),\eta(t))dt + (x(s),\eta(s)) = 0 \tag{2.8}$$

and would be given by

$$x(t) = \Phi(t,s)x(s) - \int_s^t \Phi(t,\sigma)B(\sigma)x(\sigma)d\sigma, \quad s \le t \le T \tag{2.9}$$

Since $A(t) + B(t)$ is closed for every $t \in J$, with domain $\mathcal{D}(A(t))$ and since $(A(t) + B(t))^* = A^*(t) + B^*(t)$, $x(t)$ would then also satisfy

$$\int_s^T (x(t),\eta'(t)-(A(t)+B(t))^*\eta(t))dt + (x(s),\eta(s)) = 0 \tag{2.10}$$

Proof. Let L be the operator in $L^2(J,H)$ defined by

$$Lf(t) = \int_s^t \Phi(t,\sigma)B(\sigma)f(\sigma)d\sigma .$$

L is linear and bounded. Equation (2.9) can now be written as an equation in $L^2(J,H)$, namely

$$x + Lx = g_s(t) \tag{2.11}$$

where we have set $g_s(t) = \Phi(t,s)x(s)$. Since

$$\int_s^T\int_s^t \|\Phi(t,\sigma)B(\sigma)\|^2 d\sigma dt < \infty$$

Tricomi's argument, [3 ;p103], can be used to conclude that L is quasi-nilpotent. In fact, for $n \geq 2$

$$(L^n f)(t) = \int_s^t M_n(t,\sigma)f(\sigma)d\sigma \tag{2.12}$$

where $M_n(t,\sigma) = \int_\sigma^t M_1(t,\tau)M_{n-1}(\tau,\sigma)d\tau$ and $M_1(t,s) = \Phi(t,s)B(s)$.

For the existence of $M_n(t,\sigma)$ we refer to [17;Vol.I,p117] and notice that $M_1(t,\sigma)$ is measurable and bounded almost everywhere.

In order to estimate the norm of $M_n(t,\sigma)$ we set

$$V(\sigma) = \int_\sigma^T \|\Phi(t,\sigma)B(\sigma)\|^2 dt.$$

By induction and with Hölder's inequality, one obtains by standard arguments

$$\|M_n(t,\sigma)\|^2 \leq 2((MN)^2/2)^{n-1}[(t-\sigma)^{2n-3}/(n-2)!]V(\sigma) \tag{2.13}$$

Therefore by virtue of (2.12) we arrive at

$$\|L^n\|^2 \leq ((MN)^2/2)^n T^{2n}/(n-1)!. \tag{2.14}$$

Since $n^n e^{-n+1} \leq n!$, it follows readily that $(\|L^n\|)^{1/n} \to 0$ as $n \to \infty$.

L being quasinilpotent enables us to prove the *uniqueness* of the weak solution. If there are two solutions of the required type, the difference $z(t)$ will satisfy $z(s) = 0$ and

$$\int_s^T (z(t),\eta'(t)-A^*(t)\eta(t))dt + \int_s^T(-B(t)z(t),\eta(t))dt = 0 .$$

Since $B(t)z(t)$ is a given function in $L^2(J,H)$, by (C.2) the above equation has a unique solution

$$z(t) = -\int_s^t \Phi(t,\sigma)B(\sigma)z(\sigma)d\sigma$$

which is continuous. Hence $z + Lz = 0$. But since $-1 \in \rho(L)$ $z(t) = 0$ for almost all $t \in J$, and since $z(t)$ is given to be continuous, so $z(t) = 0$ for every $t \in J$. This proves the uniqueness.

The *existence* of a solution: Since the spectral radius of L is zero, we will use a Neumann expansion for $(-I - L)^{-1}$. In view of (2.11) we define

$$x(t) = -\Sigma_0^\infty(-1)^{n+1}L^n h_s(t) = h_s(t) - \Sigma_1^\infty(-1)^{n+1}L^n h_s(t) \tag{2.15}$$

where we have set

$$h_s(t) = \Phi(t,s)x(s) + \int_s^t \Phi(t,\sigma)f(\sigma)d\sigma,$$

for some $f \in L^2(J,H)$.

The function $x(t)$ is continuous in $[s,T]$ because

$$\|x(t_1)-x(t_2)\| \le \|h_s(t_1)-h_s(t_2)\|\,[1 + \Sigma_1^\infty\|L^n\|] \to 0 \quad \text{as} \quad |t_1-t_2| \to 0,$$

since by (2.14) $\Sigma_1^\infty\|L^n\| < \infty$ and $h_s(t)$ is continuous in $[s,T]$.

Next we set

$$K_n(t,\sigma) = \Sigma_{j=1}^n(-1)^{j+1}M_j(t,\sigma) \quad .$$

The sequence of linear operators in $L^2(J,H)$ defined by

$$Q_n = \int_s^t K_n(t,\sigma)\cdot d\sigma$$

is convergent in $\mathcal{B}(L^2(J,H))$. To show this, let f be in $L^2(J,H)$, then

$$\|Qf - Q_n f\|^2 \le \int_s^T\Big(\int_s^t\|(\Sigma_{j=n+1}^\infty(-1)^{j+1}M_j(t,\sigma))f(\sigma)\|d\sigma\Big)^2 dt$$

$$\le \|f\|^2\int_s^T\int_s^t(\Sigma_{j=n+1}^\infty\|M_j(t,\sigma)\|)^2 d\sigma dt\ .$$

By (2.13) $\|M_j(t,\sigma)\| \le (2V(\sigma))^{1/2}P^{j-1}/((j-2)!T)^{1/2}$ where we have set $P^2 = (MNT)^2/2$. Thus the infinite series

$$\Sigma\, P^{j-1}/((j-2)!T)^{1/2} \le C(M,N,T) \tag{2.16}$$

is convergent. It follows now, since $\int_s^t V(\sigma)d\sigma < \infty$, that Q_n converges to Q in $\mathcal{B}(L^2(J,H))$ as $n \to \infty$. Hence we have that

$$Q = \int_s^t K(t,\sigma) \cdot d\sigma$$

is a bounded linear operator in $L^2(J,H)$.

Taking this and eqn. (2.12) into account, $x(t)$ of (2.15) becomes

$$x(t) = h_s(t) - \lim_{n\to\infty} \int_s^t K_n(t,\sigma)h_s(\sigma)d\sigma = h_s(t) - \int_s^t K(t,\sigma)h_s(\sigma)d\sigma \tag{2.17}$$

where one can easily derive from the above that the convergence takes place pointwise in H also.

Note especially that by (2.16)

$$\int_s^t \|K(t,\sigma)\|\, d\sigma \le \int_s^t \Sigma_{j=1}^{\infty} \|M_j(t,\sigma)\|\, d\sigma \le 2^{1/2}C(M,N,T)\int_s^t V(\sigma)^{1/2}d\sigma$$

$$\le MNT^{3/2}\cdot C(M,N,T) \tag{2.18}$$

is uniformly bounded in $t \in J$. And moreover by the same argument it follows that Q is also quasinilpotent.

Next it is verified that $x(t)$ as given by (2.17) satisfies the equation

$$x(t) = \Phi(t,s)x(s) - \int_s^t \Phi(t,\sigma)[B(\sigma)x(\sigma) - f(\sigma)]d\sigma. \tag{2.19}$$

In fact, eqn. (2.19) can be written in the form

$$x(t) = h_s(t) - Lx(t), \tag{2.20}$$

and thus in operator notation, we have to show that

$$x(t) = (I-Q)h_s(t) = h_s(t) - L(I-Q)h_s(t).$$

Since $-1 \in \rho(L)$, this reduces to

$$Qh_s(t) = (I+L)^{-1}Lh_s(t).$$

Using the Neumann expansion for $(I+L)^{-1}$, and (2.15), the last equality is seen to be valid.

Now we show that the function defined by (2.19) satisfies

$$\int_s^T (x(t),\eta'(t)-A^*(t)\eta(t))dt + \int_s^T (f(t)-B(t)x(t),\eta(t))dt + (x(s),\eta(s)) = 0$$

for any test function $\eta(t)$, i.e. it is the weak solution of $x' + (A(t)+B(t))x = f(t)$.

In order to verify this we use (2.20) and note that $h_s(t)$ is the weak solution of $x' + A(t)x = f(t)$. Thus it remains to show that for any given test function $\eta(t)$

$$\int_s^T (Lx(t),\eta'(t)-A^*(t)\eta(t)) + (B(t)x(t),\eta(t))dt = 0.$$

In more detail the above equation becomes

$$H_\eta(Bx) = \int_s^T\int_s^t (\Phi(t,\sigma)B(\sigma)x(\sigma),\eta'(t) - A^*(t)\eta(t))d\sigma dt$$

$$+ \int_s^T (B(t)x(t),\eta(t))dt = 0.$$

Since $B(t)x(t)$ is a given function in $L^2(J,H)$ we can approximate it by continuous functions $p_n(t)$. But for a continuous function the result follows directly from [41;p114]. Since $H_\eta(p_n) \to H_\eta(Bx)(n\to\infty)$, just as in the proof of Theorem 2.1, the result is proven.

For equation (2.1) an evolution operator $\Psi(t,s)$ is now defined on the basis of (2.17) (by letting $f = 0$ in $h_s(t)$). For any $s \in [0,T)$ and any $x(s) \in H$, $s \le t \le T$,

$$\Psi(t,s)x(s) = \Phi(t,s)x(s) - \int_s^t K(t,\sigma)\Phi(\sigma,s)x(s)d\sigma \quad . \tag{2.21}$$

Clearly for any s,t, $0 \le s \le t \le T$, $\Psi(t,s)$ is a bounded linear operator in H. It satisfies $\Psi(t,t) = I$, and for $s \le \sigma \le t$ the uniqueness provides $\Psi(t,s) = \Psi(t,\sigma)\Psi(\sigma,s)$. Furthermore by (2.18) we have

$$\|\Psi(t,s)\| \le M[1 + MNT^{3/2} \cdot C(M,N,T)] \le K$$

i.e. $\Psi(t,s)$ is uniformly bounded in s and t by a constant K which depends only on M, N and the interval $J = [0,T]$.

For $f(t) = 0$ and for given $s \in [0,T)$ and $x(s) \in H$, $x(t) = \Psi(t,s)x(s)$ by definition. Thus as was shown in (2.19), $\Psi(t,s)$ satisfies (2.6). By rearranging the equation in (2.17) and replacing $h_s(\tau)$ (see (2.15)) by $\Phi(\tau,s)x(s) + \int_s^\tau \Phi(\tau,\sigma)f(\sigma)d\sigma$ therein, we can write $x(t)$ of (2.17) in the form (2.7). Thus (2.7) is the unique weak solution of $x' + (A(t) + B(t))x = f(t)$.

The strong continuity of $\Psi(t,s)$ in t, $0 \le s \le t \le T$ follows from the statement after eqn. (2.15). Now we consider the strong continuity of $\Psi(t,s)$ in s. Since $\Phi(t,s)$ is strongly continuous in $\{(t,s) \in J^2 | s \le t\}$ it is uniformly strongly continuous there. Let $s_1 \le s_2 \le t$ and $x_0 \in H$, then

$$\|\Psi(t,s_1)x_0 - \Psi(t,s_2)x_0\| \le \|\Phi(t,s_1)x_0 - \Phi(t,s_2)x_0\| +$$

$$+ \left\|\int_{s_1}^{s_2} K(t,\sigma)\Phi(\sigma,s_1)x_0 d\sigma\right\| + \int_{s_2}^t \|K(t,\sigma)\| \cdot \|\Phi(\sigma,s_1)x_0 - \Phi(\sigma,s_2)x_0\| d\sigma$$

which converges to zero if $|s_2 - s_1| \to 0$.

To prove the compactness of $\Psi(t,s)$ for $s < t$, we use (2.6) and the fact that $\Phi(t,\sigma)B(\sigma)\Psi(\sigma,s)$ is compact for almost every $\sigma \in (s,t)$. Thus for (x_n), a sequence weakly converging to x in H,

$$\Phi(t,\sigma)B(\sigma)\Psi(\sigma,s)x_n \to \Phi(t,\sigma)B(\sigma)\Psi(\sigma,s)x \quad (n\to\infty)$$

strongly for almost every $\sigma \in (s,t)$. Since (x_n) is a bounded sequence the dominated convergence theorem guarantees that $\Psi(t,s)x_n \to \Psi(t,s)x$ $(n\to\infty)$. Hence $\Psi(t,s)$ $(s<t)$ is a compact operator. This completes the proof of Theorem 2.3 □

If we use the expressions in (2.6) and (2.21) for $\Psi(t,s)$, we can prove the following result

Theorem 2.4* $\Psi(t,s)$ *is continuous on* $S' = \{(t,s) \in J^2 \mid 0 \le s < t \le T\}$ *in the uniform operator topology*.

The proof is the same as for Theorem 2.2. We only need to know that by choosing $|s_2-s_1|$ sufficiently small, we can ensure that

$$W = \int_{s_1}^{s_2} \| K(t,\sigma)\| d\sigma$$

becomes arbitrarily small. To see this one proceeds just as in (2.18), to obtain

$$W \le 2^{1/2} C(M,N,T)|s_2-s_1|^{1/2}\{\int_{s_1}^{s_2}\int_{\sigma}^{T} \|\Phi(t,\sigma)B(\sigma)\|^2 dt d\sigma\}^{1/2}$$

$$= 2^{1/2} C(M,N,T)|s_2-s_1|^{1/2}\{\int_{s_1}^{s_2}\int_{s_1}^{t} + \int_{s_2}^{T}\int_{s_1}^{s_2} \|\Phi(t,\sigma)B(\sigma)\|^2 d\sigma dt\}^{1/2} \quad \square$$

The remaining part of this section will be concerned with convergence properties of equation (2.1). The convergence results given by Becker [6] are valid also in our more general situation.

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*As remarked earlier, this result is not needed for the proof of the main theorem.

Thus Theorem 2.6 and Corollaries 2.7 and 2.8 are taken directly from [6; Theorem 2.2, Cor. 2.4 and 2.5]. The proofs are also the same except for minor modifications.

Because of the weak compactness property of a Hilbert space and the following adaptation of Proposition 1.2, the weak convergence of the perturbation operator $B(t)$ already implies the strong convergence of the evolution operator $\Psi(t,s)$. We conclude the section with Proposition 2.9 which concerns the convergence of periodic solutions of eqn. (3.5).

Proposition 2.5. *Let* $\Phi(t,s)$ *be an evolution operator of* (2.1) *satisfying* (C.1) - (C.3) *of* Definition 2.1. *Then if* $J_S = [s,T]$,

(a) *The map* $L:L^2(J_S,H) \to H$ *defined by* $x(t) \to \int_s^t \Phi(t,\sigma)x(\sigma)d\sigma$ *is compact for each* $s \in J$ *and each* $t > s$, $t \in J_S$.

(b) *The map in* $L^2(J_S,H)$ *defined by* $x(t) \to \int_s^t \Phi(t,\sigma)x(\sigma)d\sigma$ *is compact for each* $s \in J$.

Theorem 2.6. *Let* $x' + A(t)x = 0$ *be of compact type and* $B_n(t)$ *a sequence in* $M(J,H)$, *satisfying* $\|B_n(t)\| \leq N$ *for almost every* $t \in J$, *where* N *is independent of* n. *Further let* $\tilde{B}_n \to \tilde{B}$ *weakly in* $B(L^2(J,H))$. *Then if* $\Phi_n(t,s)$ *denotes the evolution operator of*

$$x' + (A(t) + B_n(t))x = 0$$

we have $\Phi_n(t,s) \to \Psi(t,s)$ *strongly, where* $\Psi(t,s)$ *is the evolution operator of* (2.6).

Proof. By (2.6) we have (s fixed)

$$\Phi_n(t,s)x(s) = \Phi(t,s)x(s) - \int_s^t \Phi(t,\tau)B_n(\tau)\Phi_n(\tau,s)x(s)d\tau \tag{2.22}$$

By the remark following Proposition 1.4, we can write (2.22) in the form $y_n = Mx - L(C_n y_n)$ in $L^2([s,T],H)$. By assumption and Theorem 2.3, $\Phi_n(t,s)$ is uniformly bounded in n, s and t $(0 \le s \le t \le T)$. Thus (y_n) is a bounded sequence and Proposition 1.4 implies that $y_n = \Phi_n(t,s)x(s)$ has a subsequence $\Phi_\ell(t,s)x(s)$ converging strongly in $L^2([s,T],H)$ to $\Psi_1(t,s)x(s)$ (say).

Now $\tilde{B}_n$ being uniformly bounded and convergent in the weak operator topology on $\mathcal{B}(L^2([s,T],H))$, and $\Phi_\ell(t,s)x(s)$ converging strongly, implies that $B_\ell(t)\Phi_\ell(t,s)x(s)$ converges weakly in $L^2([s,T],H)$ to $B(t)\Psi_1(t,s)x(s)$. Hence in (2.22)

$$\int_s^t \Phi(t,\tau)[B_\ell(\tau)\Phi_\ell(\tau,s) - B(\tau)\Psi_1(\tau,s)]x(s)d\tau \to 0$$

strongly in $L^2([s,T],H)$. Taking limits $(\ell\to\infty)$ in (2.22) we see that $\Psi_1(t,s)$ satisfies the integral equation (2.6) and by the uniqueness, $\Psi_1(t,s) = \Psi(t,s)$.

We have also convergence in H. For, subtracting (2.22) from (2.6) and fixing t we obtain

$$(\Psi(t,s) - \Phi_\ell(t,s))x(s) = \int_s^t \Phi(t,\tau)B_\ell(\tau)(\Phi_\ell(\tau,s) - \Psi(\tau,s))x(s)d\tau$$

$$+ \int_s^t \Phi(t,\tau)(B_\ell(\tau) - B(\tau))\Psi(\tau,s)x(s)d\tau .$$

Because $\|\Phi(t,\tau)B_\ell(\tau)\| \le MN$, the first integral converges to zero strongly in H.

Since $(B_n(t) - B(t))\Psi(t,s)x(s)$ converges weakly in $L^2([s,T],H)$ to zero, Proposition 2.5 (a) implies that for fixed t, the second integral converges strongly in H to zero.

Since every sequence of values of n has a subsequence ℓ for which $\Phi_\ell \to \Psi$, it follows that the whole sequence converges to Ψ $\square$

Corollary 2.7. *Under the hypotheses of* Theorem 2.6, *if* (y_n) *is a bounded sequence in* H, *then for each* $t > s$, $\Phi_n(t,s)y_n$ *has a strongly convergent subsequence in H.*

Proof.
$$\Phi_n(t,s)y_n = \Phi(t,s)y_n - \int_s^t \Phi(t,\tau)B_n(\tau)\Phi_n(\tau,s)y_n d\tau \quad . \tag{2.23}$$

Since (y_n) and $(B_n(t)\Phi_n(t,s)y_n)$ are uniformly bounded, and since the operator $\Phi(t,s)$ and the integral operator $\int_s^t \Phi(t,\tau)\cdot d\tau$ are compact (see Prop. 2.5), the righthand side of (2.23) has a subsequence which converges strongly in H.□

Corollary 2.8. *Under the hypotheses of* Theorem 2.6, *if* λ *belongs to the resolvent set of* $\Phi_n(t,s)$ *for all* $n \in \mathbb{N}$ ($t > s$ *fixed*) *and also to that of* $\Psi(t,s)$, *then* $(\lambda I - \Phi_n(t,s))^{-1}$ *converges strongly to* $(\lambda I - \Psi(t,s))^{-1}$ (*in* H).

Proof. Observe that $(\lambda I - \Phi_n(t,s))^{-1}x - (\lambda I - \Psi(t,s))^{-1}x$

$$= (\lambda I - \Phi_n(t,s))^{-1}(\Phi_n(t,s) - \Psi(t,s))(\lambda I - \Psi(t,s))^{-1}x. \tag{2.24}$$

Since $\Phi_n(t,s) \to \Psi(t,s)$ strongly, it suffices to show that for fixed s and t with $t > s$, $(\lambda I - \Phi_n(t,s))^{-1}$ is uniformly bounded. If this does not hold, we can find a sequence (x_n) with $\|x_n\| = 1$ such that $y_n = (\lambda I - \Phi_n(t,s))^{-1}x_n$ satisfies $\lim_{n\to\infty}\|y_n\| = \infty$.
Setting $z_n = y_n/\|y_n\|$ and $w_n = x_n/\|y_n\|$ we see that

$$\Phi_n(t,s)z_n = \lambda z_n - w_n \tag{2.25}$$

and $\|z_n\| = 1$, $\lim_{n\to\infty}\|w_n\| = 0$.

By Corollary 2.7, there is a subsequence $\Phi_\ell(t,s)z_\ell$ that is convergent.

Thus by (2.25), (z_ℓ) converges to z say. Now taking limits in (2.25) we see that

$$\Psi(t,s)z = \lambda z \quad \text{and} \quad \|z\| = 1$$

because $\Phi_n(t,s)$ is uniformly bounded. This contradicts the fact that λ belongs to the resolvent set of $\Psi(t,s)$, thereby proving the Corollary □

In view of Theorem 3.1 and II Theorem 2.1, where we will have to verify the continuity of a certain operator G, we prove here the following

Proposition 2.9. *Let* $(\Phi_n(t,s))$ *be a sequence of evolution operators satisfying* (C.1) *of* Definition 2.1 *and converging strongly in* H *to an evolution operator* $\Phi(t,s)$ *, for* $0 \le s < t \le T$ *. We assume that* $\Phi_n(t,s)$ *is bounded independently of* n, s *and* t *,* $(t,s) \in S = \{(t,s) | 0 \le s \le t \le T\}$ *. Suppose that* λ *belongs to the resolvent set of* $\Phi_n(T,0)$ *for all* $n \in \mathbb{N}$ *and also to that of* $\Phi(T,0)$ *and assume that* $(\lambda I - \Phi_n(T,0))^{-1}$ *is a bounded sequence. Assume finally that the sequence* (f_n) *converges to* f *in* $L^2(J,H)$ *. Let* x_n *in* H *be defined by*

$$x_n = (\lambda I - \Phi_n(T,0))^{-1} \int_0^T \Phi_n(T,\tau) f_n(\tau) d\tau$$

and $\psi_n(t)$ *in* $L^2(J,H)$ *by*

$$\psi_n(t) = \Phi_n(t,0)x_n + \int_0^t \Phi_n(t,\tau) f_n(\tau) d\tau \ .$$

It then follows that

(i) $$x_n \to x = (\lambda I - \Phi(T,0))^{-1} \int_0^T \Phi(T,\tau) f(\tau) d\tau$$

in H *as* $n \to \infty$ *, and*

(ii) $$\psi_n(t) \to \psi(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau$$

in $L^2(J,H)$, *(and* H) , *as* $n \to \infty$.

Remark. The strong continuity and the strong convergence of $\Phi_n(t,s)$ do not imply that $\Phi_n(t,s)$ is uniformly bounded in n , t and s . However if we suppose that $\Phi_n(t,s)$ is strongly equicontinuous, an application of the Banach-Steinhaus theorem will yield such a bound. (We note that S is compact).

Proof. (i) We first prove the convergence of x_n ,

$$x - x_n = \{(\lambda I - \Phi(T,0))^{-1} - (\lambda I - \Phi_n(T,0))^{-1}\}\int_0^T \Phi(T,\tau)f(\tau)d\tau$$

$$+ (\lambda I-\Phi_n(T,0))^{-1}\{\int_0^T [\Phi(T,\tau)-\Phi_n(T,\tau)]f(\tau)d\tau + \int_0^T \Phi_n(T,\tau)[f(\tau)-f_n(\tau)]d\tau\}$$

$$\equiv I + II .$$

By virtue of Corollary 2.8, see (2.24), and our assumptions, the first term I converges to zero in H as $n \to \infty$. As regards the second term II, it remains to show that

$$\int_0^T [\Phi(T,\tau) - \Phi_n(T,\tau)]f(\tau)d\tau \quad \text{and} \quad \int_0^T \Phi_n(T,\tau)[f(\tau) - f_n(\tau)]d\tau$$

converge to zero as $n \to \infty$, because $(\lambda I - \Phi_n(T,0))^{-1}$ is bounded. Using the assumptions and the dominated convergence theorem this follows immediately since $\Phi_n(t,s)$ is uniformly bounded in n , s and t .

(ii) Concerning the convergence of $\psi_n(t)$, consider

$$\psi_n(t) - \psi(t) = \Phi_n(t,0)[x_n-x] + [\Phi_n(t,0) - \Phi(t,0)]x \;+$$

$$+ \int_0^t \Phi_n(t,\tau)[f_n(\tau) - f(\tau)]d\tau + \int_0^t [\Phi_n(t,\tau) - \Phi(t,\tau)]f(\tau)d\tau$$

$$\equiv I + II + III + IV .$$

We will show that each of the four terms on the right hand side of the above equation converges to zero in $L^2(J,H)$. The first term I converges to zero in H , $(n\to\infty)$ by (i) and because $\Phi_n(t,s)$ is uniformly bounded. Hence it converges also in $L^2(J,H)$. Term II approaches zero in $L^2(J,H)$, $(n\to\infty)$ by assumption and an application of the dominated convergence theorem.

The norm in $L^2(J,H)$ of the expression in III is bounded by $CT\|f_n - f\|$, where $\|\Phi_n(t,s)\| \le C$, and therefore approaches zero $(n\to\infty)$, by assumption.

Concerning the last term IV, we set

$$g_n(t) = \int_0^t \|[\Phi_n(t,\tau) - \Phi(t,\tau)]f(\tau)\|d\tau \quad \text{for} \quad t > 0 .$$

The integrand here, converges to zero for almost every $\tau < t$, and it is bounded by $2C\|f(\tau)\|$. Therefore $g_n(t) \to 0$ $(n\to\infty)$, for any $t > 0$ by the dominated convergence theorem. Since by Hōlder's inequality

$$|g_n(t)|^2 \le 4C^2 t\|f\|^2 ,$$

it follows by yet another application of the dominated convergence theorem that the last term IV tends to zero in $L^2(J,H)$ as $n \to \infty$. This completes the proof of Proposition 2.9 □ .

The results which we have proved will enable us now to establish the main theorem (Theorem 3.1).

3. PERIODIC SOLUTIONS OF A SEMILINEAR EQUATION OF EVOLUTION

In this section we will prove by an application of Schauder's fixed point theorem, that the semilinear equation of evolution

$$x' + (A(t) + B(t,x))x = f(t,x) \tag{3.1}$$

has a weak, periodic solution.

Theorem 3.1. *Let* $x' + A(t)x = 0$ *be of compact type. Let* $B(t,x)$ *be in* $B(H)$, *strongly measurable in* t *for each* $x \in H$, *and strongly continuous in* x, *for almost every* $t \in J$. *Assume that* $B(t,\mu(t))$ *lies in a weakly closed subset* S *of* S_N *(see Prop. 1.1), for each* $\mu(t) \in L^2(J,H)$. *Suppose further that for all* $\tilde{B}$ *in* S *the equation*

$$x' + (A(t) + B(t))x = 0 \tag{3.2}$$

has unique weak, periodic solution $x = 0$.
Concerning $f(t,x): J \times H \to H$ *we assume that (1) it is measurable in* t *for each* $x \in H$ *and continuous in* x *for almost every* $t \in J$.
(2) For every $x \in L^2(J,H)$, $f(t,x(t)) \in L^2(J,H)$.
(3) For every sequence (x_ℓ) *in* $L^2(J,H)$ *with* $\|x_\ell\| \le \ell$

$$\liminf_{\ell \to \infty} (1/\ell) \int_0^T \|f(t,x_\ell(t))\| dt = 0. \tag{3.3}$$

Then equation (3.1) *has a weak, periodic solution, i.e. a weak solution satisfying* $x(0) = x(T)$.

Remark. The conditions on the perturbation $B(t,x)$ are the same as in [6 ;Theorem 3.1]. The asymptotic sublinearity assumption made on $f(t,x)$ is somewhat simpler than that used by Opial [62;p588] and Ward [76;p471], yet the same method will be employed to show that there exists a ball which is mapped into itself by a compact transformation.

In preparation for the proof of Theorem 3.1, we relate the notion of a weak solution of (3.1) to a non-linear integral equation.

Proposition 3.2. *Assume that the hypotheses of Theorem* 3.1 *hold and suppose that* $y(t)$ *is a continuous function on* J *which satisfies*

$$y(t) = \Phi(t,0)x_0 - \int_0^t \Phi(t,\tau)[B(\tau,y(\tau))y(\tau) - f(\tau,y(\tau))]d\tau \tag{3.4}$$

for some $x_0 \in H$, *where* $\Phi(t,s)$ *denotes the evolution operator of* $x' + A(t)x = 0$. *The function* $y(t)$ *is then a weak solution of* (3.1) *satisfying an initial condition.*

Proof. By assumption and using the proof of Theorem 2.3, (we note (2.19)), we have that for any $\mu(t)$ in $L^2(J,H)$ the function satisfying

$$\psi(t) = \Phi(t,0)x_0 - \int_0^t \Phi(t,\tau)[B(\tau,\mu(\tau))\psi(\tau) - f(\tau,\mu(\tau))]d\tau$$

is the unique weak solution of

$$x' + (A(t) + B(t,\mu(t)))x = f(t,\mu(t))$$

with $\psi(0) = x_0$. Define the map G' on $L^2(J,H)$ by $G'(\mu) = \psi$. Any continuous function $y(t)$ which satisfies (3.4) is a fixed point of G' and thus a weak solution of (3.1) with $y(0) = x_0$.

We can also argue as follows. Suppose $y(t)$ is given as in the assumptions, thus $B(t,y(t))$ and $f(t,y(t))$ are determined. Hence by the method used in the proof of Theorem 2.3, (after (2.20)), one can verify directly that $y(t)$ is a weak solution of (3.1) $\square$.

<u>Proof of Theorem 3.1</u>. We apply the Schauder fixed point theorem. Firstly we show that for $h(t) \in L^2(J,H)$ and for any $B(t)$ as given in the statement, the equation

$$x' + (A(t) + B(t))x = h(t) \tag{3.5}$$

has a unique weak, periodic solution $y(t)$.

Let $\Psi(t,s)$ denote the evolution operator of (3.2) as guaranteed by Theorem 2.3. Then there exists a periodic solution $y(t)$ of (3.5) if and only if there exists $x_0 \in H$ such that

$$\begin{aligned} y(t) &= \Psi(t,0)x_0 + \int_0^t \Psi(t,\tau)h(\tau)d\tau \\ \text{and} \qquad (I - \Psi(T,0))x_0 &= \int_0^T \Psi(T,\tau)h(\tau)d\tau \quad . \end{aligned} \tag{3.6}$$

By the uniqueness assumption, there does not exist $x_0 \neq 0$ such that $(I - \Psi(T,0))x_0 = 0$. Thus by the compactness of $\Psi(T,0)$, $(I - \Psi(T,0))$ is invertible, i.e. 1 lies in the resolvent set of $\Psi(T,0)$.

We now show that for any $B(t)$ in S, $(I - \Psi(T,0))^{-1}$ is bounded by a constant depending only on N and $A(t)$. If not, there is a sequence $(B_n(t))$ in S such that $(I - \Phi_n(T,0))^{-1}$ is unbounded, where $\Phi_n(t,s)$ is the evolution operator corresponding to $B_n(t)$. By Proposition 1.1, there is then a $B^0(t) \in S$ and a subsequence ℓ such that $B_\ell(t) \to B^0(t)$ weakly in $\mathcal{B}(L^2(J,H))$. Since $B^0(t) \in S$ and because of the uniqueness hypothesis, $(I - \Psi^0(T,0))$ is invertible ($\Psi^0(t,s)$ is the evolution operator corresp. to $B^0(t)$). By Corollary 2.8 and the principle of uniform boundedness $(I - \Phi_\ell(T,0))^{-1}$ is uniformly bounded. The same argument shows that any subsequence contains a bounded subsequence, which gives a contradiction. Hence $(I - \Psi(T,0))^{-1}$ is uniformly bounded. Because of the above result which is due to Becker [6; p42], we do not need a priori estimates of the derivative.

Given $\rho(t) \in L^2(J,H)$. We define the map $G: L^2(J,H) \to L^2(J,H)$ by $G\rho = \psi$, where ψ is the unique weak, periodic solution of

$$x' + (A(t) + B(t,\rho(t)))x = f(t,\rho(t)) \quad . \tag{3.7}$$

Then by (C.2) of Definition 2.1 (see also (2.19)) applied to $x' + A(t)x = 0$,

$$G\rho(t) = \psi(t) = \Phi(t,0)x_0 - \int_0^t \Phi(t,\tau)[B(\tau,\rho(\tau))\psi(\tau) - f(\tau,\rho(\tau))]d\tau \quad . \tag{3.8}$$

We will show that (a) G is compact, (b) G is continuous, (c) there exists a ball K_n in $L^2(J,H)$ such that $G(K_n) \subset K_n$.

Assuming these, by Schauder's theorem, G has a fixed point $y(t)$. The continuity of $y(t)$ is then easily established. It follows that $y(t)$ satisfies (3.4) and hence by Proposition 3.2, $y(t)$ is a weak, periodic solution of (3.1), thereby proving the theorem.

(a) G *is compact*. Let (ρ_n) be a bounded sequence in $L^2(J,H)$. Proposition 1.3 then implies that $(f(t,\rho_n(t)))$ is also bounded in $L^2(J,H)$. We show first that (x_n) and (ψ_n) are bounded in H and $L^2(J,H)$ respectively, where x_n denotes the initial value corresponding to ψ_n. Denote by $\Phi_n(t,s)$ the evolution operator of

$$x' + (A(t) + B(t,\rho_n(t)))x = 0$$

which is guaranteed by Theorem 2.3. Then by the analogue of (3.6)

$$\psi_n(t) = \Phi_n(t,0)x_n + \int_0^t \Phi_n(t,\tau)f(\tau,\rho_n(\tau))d\tau \tag{3.9}$$

$$x_n = (I - \Phi_n(T,0))^{-1}\int_0^T \Phi_n(T,\tau)f(\tau,\rho_n(\tau))d\tau \ . \tag{3.10}$$

Since $\|B(t,\rho_n(t))\| \le N$ independently of n and for almost every $t \in J$, we have by Theorem 2.3, $\|\Phi_n(t,s)\| \le K$ independently of n. By what was shown previously $\|(I - \Phi_n(T,0))^{-1}\| \le K'$. Thus it follows immediately that (x_n) is a bounded sequence in H and $(\psi_n(t))$ is uniformly bounded in H, in n and $t \in J$. We conclude that the expression

$$[B(t,\rho_n(t))\psi_n(t) - f(t,\rho_n(t))]$$

in (3.8) is bounded in $L^2(J,H)$.

In view of Proposition 1.4 and the remark following it, the analogue of (3.8) for ρ_n, can be written as

$$y_n = Mx_n - L(C_n y_n - f_n)$$

in $L^2(J,H)$. By virtue of Propositions 1.4 and 2.5, $(y_n) = (\psi_n(t)) = (G\rho_n)$ has a convergent subsequence in $L^2(J,H)$, i.e. G is compact.

(b) G *is continuous*. Let $\rho_n \to \rho$ in $L^2(J,H)$. By Proposition 1.3, $f(t,\rho_n(t)) \to f(t,\rho(t))$ in $L^2(J,H)$. For any fixed $y(t) \in L^2(J,H)$, Proposition 1.3 also applies to $B(t,x)y(t)$. Hence $B(t,\rho_n(t))y(t) \to B(t,\rho(t))y(t)$ in $L^2(J,H)$. And thus $\tilde{B}(t,\rho_n(t))$ converges strongly to $\tilde{B}(t,\rho(t))$ in $\mathcal{B}(L^2(J,H))$. If $\Phi_n(t,s)$ denotes the evolution operator corresponding to $B(t,\rho_n(t))$, then by Theorem 2.6, $\Phi_n(t,s)$ is strongly convergent in $L^2(J,H)$ and in H, and is uniformly bounded.

Further it follows by what we have shown previously that the remaining assumptions of Proposition 2.9 are satisfied for $\lambda = 1$. Hence applying this proposition we conclude that $\psi_n(t)$ as given by (3.9) and (3.10) converges to $\psi(t) = G\rho(t)$ in $L^2(J,H)$, i.e. G is continuous.

(c) *Letting* $K_n = \{x \in L^2(J,H) \mid \|x\| \le n\}$, *there exists an* n *such that* $G(K_n) \subset K_n$. Suppose this is not so. Hence there exists a sequence (x_ℓ) in $L^2(J,H)$ such that $x_\ell \in K_\ell$ and $Gx_\ell \notin K_\ell$, i.e. $1 < \ell^{-2}\|Gx_\ell\|^2$ for all $\ell \in \mathbb{N}$.

Let $\Phi_\ell(t,s)$ denote the evolution operator of $x' + (A(t) + B(t,x_\ell(t)))x = 0$. By Theorem 2.3 and our assumption, $\|\Phi_\ell(t,s)\| \le K$ independently of ℓ and by what was proved previously, $\|(I - \Phi_\ell(T,0))^{-1}\| \le K'$. Combining (3.9) and (3.10) we obtain

$$Gx_\ell(t) = \Phi_\ell(t,0)(I - \Phi_\ell(T,0))^{-1}\int_0^T \Phi_\ell(T,\tau)f(\tau,x_\ell(\tau))d\tau$$

$$+ \int_0^t \Phi_\ell(t,\tau)f(\tau,x_\ell(\tau))d\tau$$

and hence

$$\|Gx_\ell(t)\|_H \le K^2K'\int_0^T \|f(\tau,x_\ell(\tau))\|d\tau + K\int_0^t \|f(\tau,x_\ell(\tau))\|d\tau$$

$$\le C\int_0^T \|f(\tau,x_\ell(\tau))\|d\tau \ .$$

Therefore for all ℓ

$$1 < \ell^{-2}\|Gx_\ell\|^2_{L^2(J,H)} \le C^2\ell^{-2}\int_0^T (\int_0^T \|f(\tau,x_\ell(\tau))\|d\tau)^2 dt$$

$$\le TC^2\{\ell^{-1}\int_0^T \|f(\tau,x_\ell(\tau))\|d\tau\}^2 .$$

which contradicts (3.3) .

Since G satisfies (a), (b), (c) we know that it has a fixed point $y(t)$. To prove the continuity of $y(t)$ in J we use (3.9). Let $s < t$, then

$$y(t) - y(s) = [\Phi_y(t,0) - \Phi_y(s,0)]x_y + \int_s^t \Phi_y(t,\tau)f(\tau,y(\tau))d\tau$$

$$+ \int_0^s [\Phi_y(t,\tau) - \Phi_y(s,\tau)]f(\tau,y(\tau))d\tau .$$

The first two terms on the right-hand side of the above equality clearly converge to zero, as $|t-s| \to 0$, in H . Concerning the third term, we let s be fixed first. The strong continuity of $\Phi_y(t,s)$ and dominated convergence imply that

$$\int_0^s [\Phi_y(t,\tau) - \Phi_y(s,\tau)]f(\tau,y(\tau))d\tau \to 0$$

as $t \to s$, in H . Now let t be fixed and define the function

$$F(\tau,s) = \|[\Phi_y(t,\tau) - \Phi_y(s,\tau)]f(\tau,y(\tau))\| \quad \text{if} \quad 0 \le \tau \le s$$

$$= 0 \qquad \text{if} \quad s < \tau \le t$$

Thus

$$\int_0^s \|[\Phi_y(t,\tau) - \Phi_y(s,\tau)]f(\tau,y(\tau))\| d\tau = \int_0^t F(\tau,s)d\tau$$

and $F(\tau,s)$ is bounded by $2K\|f(t,y(t))\|$ which is integrable. In order to apply the dominated convergence theorem it remains to show that

$$\lim_{s\to t} F(\tau,s) = 0$$

for almost every $\tau \in (0,t)$. Since τ remains fixed while s approaches t , we may assume that $\tau < s < t$ for almost every $\tau \in (0,t)$. Therefore

$$F(\tau,s) = \|[\Phi_y(t,\tau) - \Phi_y(s,\tau)]f(\tau,y(\tau))\|$$

and hence does approach zero as $s \to t$, for almost every $\tau \in (0,t)$. Hence $y(t)$ is continuous in J. This completes the proof that (3.1) has a weak, periodic solution $\square$.

4. EXAMPLE

We illustrate the results of Section 3 by the following example. Here we adopt the notation of Tanabe [72]. The section consists of three parts. First a strongly elliptic partial differential operator $A(t)$ whose domain varies in t is defined, following [72;p140]. We prove that its inverse is compact. Secondly, the non-linear perturbation $B(t,x)$ is considered, and conditions are given so that the uniqueness assumption of Theorem 3.1 is satisfied. These conditions are based on Landesman and Lazer [44] and are similar to [6; Theorem 4.1], but here we need not assume that A is self-adjoint. Lastly an example of a sublinear function $f(t,x)$ is given as described in Theorem 3.1.

Denote with $H_m(\Omega)$, the set of all functions which, together with their derivatives to order m in the sense of distribution, belong to the Lebesgue space $L^2(\Omega)$, where Ω is a bounded region of class C^m in $\mathbb{R}^n$. We denote the norm of $f \in H_m(\Omega)$ by $\|f\|_m$, c.f. [72;p11-14]. Let the operator defined in Ω by

$$A(x,t,D) = \sum_{|\alpha|\leq 2p} a_\alpha(x,t)D^\alpha$$

be strongly elliptic, uniformly in $t \in J = [0,T]$, c.f.[72;p77,88]. The following assumptions are made about the coefficients $a_\alpha(x,t)$. For each $t \in J$, $a_\alpha(x,t)$, $|\alpha| = 2p$ are continuous in $\bar{\Omega}$, and $a_\alpha(x,t)$, $|\alpha| < 2p$, are bounded and measurable in Ω. Every coefficient is assumed to satisfy a Hölder condition of order h uniformly in t, namely

$$\max_{|\alpha|\leq 2p} \sup_{x\in\Omega} |a_\alpha(x,t) - a_\alpha(x,s)| \leq L|t-s|^h .$$

Further for each $t \in J$ let

$$B_j(x,t,D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x,t)D^\beta \quad (j=1,\ldots p)$$

be a linear differential operator of order $m_j < 2p$ independently of t, where the coefficients $b_{j\beta}$ are defined in $\partial\Omega$ and belong to C^{2p-m_j} for any $t \in [0,T]$.

Moreover assume that these boundary differential operators are normal c.f. [72;p79].

We can find now an angle $\theta_0 \in [0,\pi/2)$ such that $\arg\mathring{A}(x,t,\xi) \neq \theta$ for all t, x and $\xi \neq 0$ and $\theta \in [\theta_0, 2\pi-\theta_0]$ (see [72;p88]). We assume that for each $t \in J$ and $\theta \in [\theta_0, 2\pi-\theta_0]$ the half-line $\arg\lambda = \theta$ is the ray of minimal growth of the resolvent of $A(t)$, where the operator $-A(t)$ is defined by

$$\mathcal{D}(A(t)) = \{u \in H_{2p}(\Omega) \,|\, B_j(x,t,D)u(x) = 0,\ x \in \partial\Omega, j=1,\ldots p\} \tag{4.1}$$

and

$$A(t)u(x) = A(x,t,D)u(x) \quad \text{for} \quad u \in \mathcal{D}(A(t)).$$

For each $t \in J$, $-A(t)$ then generates an analytic semigroup in $L^2(\Omega)$, (see [72;p83,88,140]).

Furthermore, if all the coefficients of $A(x,t,D)$ and $B_j(x,t,D)$ $(j=1,\ldots p)$ are differentiable in t and these derivatives with respect to t are Höldercontinuous uniformly in t, then Tanabe shows [72;p141-144] that his assumptions 5.1.1, 5.3.1 - 5.3.3 (see [72; p117,129,130]) are satisfied, i.e. our hypotheses (AI.1) - (AI.3) hold.

We may suppose that $0 \in \rho(A(t))$ by adding, if necessary, a sufficiently large positive number to $A(t)$. By an application of Rellich's Lemma (see e.g. [72;p14]) one can conclude that $A^{-1}(t)$ is a compact operator in $L^2(\Omega)$ for any $t \in J$. For a given $t \in J$, $A^{-1}(t)$ maps $L^2(\Omega)$ into the domain of $A(t)$. And the solution $u \in H_{2p}(\Omega)$ of

$$A(x,t,D)u(x) = f(x)$$

and $\quad B_j(x,t,D)u(x) = 0, \; x \in \partial\Omega, \; j=1,\dots p,$

for any given $f \in L^2(\Omega)$, satisfies the estimate

$$\|u\|_{2p} = \|A^{-1}(t)f\|_{2p} \leq C\|f\|_{L^2(\Omega)}$$

by the closed-graph theorem. Hence $A^{-1}(t): L^2(\Omega) \to H_{2p}(\Omega)$ is bounded, whereas the identity transformation $H_{2p}(\Omega) \to L^2(\Omega)$ is compact. It follows that our hypothesis (AI.4) holds. Thus the operator $A(t)$ as defined in (4.1) satisfies all our assumptions (AI.1) - (AI.4). In the following we set $H = L^2(\Omega)$.

Next we will consider the perturbation $B(t,x)$ and show that the uniqueness assumption of Theorem 3.1 is fulfilled if we let

$$S = \{C(t) \in M(J,H) \,|\, \|\mu I + C(t)\| \leq \alpha, \; \mu \in \mathbb{C}, \; \alpha > 0\} \tag{4.2}$$

and if we choose α and μ suitably. The set $S \subset S_N$ $(N = \alpha + |\mu|)$ is the translation of the weakly compact set S_α and is itself weakly compact by the continuity of translation, see [6;p44].
By [17;Vol.I,VI.9.2] S is also closed in the weak topology in $B(L^2(J,H))$.
We will show now that we can find μ and α such that the equation

$$x' + (A(t) + C(t))x = 0 \tag{4.3}$$

admits only the zero periodic solution for any $C(t) \in S$. We write eqn. (4.3) in the form

$$x' + (A(t) - \mu I)x = -(C(t) + \mu I)x \quad . \tag{4.4}$$

If $\Phi(t,s)$ denotes the evolution operator of $x' + A(t)x = 0$, then it follows that $e^{\mu(t-s)}\Phi(t,s)$ is the evolution operator of

$$x' + (A(t) - \mu I)x = 0 \quad . \tag{4.5}$$

Provided that $e^{-\mu T}$ does not coincide with the eigenvalues of $\Phi(T,0)$, eqn. (4.5) can only have the zero periodic solution. In fact such a solution $y(t) = e^{\mu t}\Phi(t,0)x$ would satisfy $(I - e^{\mu T}\Phi(T,0))x = 0$. Denote the eigenvalues of the compact operator $\Phi(T,0)$ by λ_i $(i \in \mathbb{N})$. Our assumption implies that $e^{\mu T}\Phi(T,0)x \neq x$ for all $x \in H$, $x \neq 0$. We can ensure that $e^{-\mu T} \neq \lambda_i$ for all $i \in \mathbb{N}$ by taking $\mu \in \mathbb{R}$, $\mu \neq -T^{-1}\log|\lambda_i|$, e.g. $\mu < -T^{-1}\log\|\Phi(T,0)\|$, since $|\lambda_i| \leq \|\Phi(T,0)\|$. The compactness of $\Phi(T,0)$ now implies that for such μ, $(I - e^{\mu T}\Phi(T,0))$ is invertible.

Hence just as in (3.6) we can conclude that the unique weak, periodic solution of $x' + (A(t) - \mu I)x = h(t)$, for some $h(t) \in L^2(J,H)$, is given by

$$x(t) = \int_0^T G_\mu(t,\tau)h(\tau)d\tau \tag{4.6}$$

where

$$\begin{aligned} G_\mu(t,\tau) &= \exp(\mu(T+t-\tau))\Phi(t,0)(I-\exp(\mu T)\Phi(T,0))^{-1}\Phi(T,\tau) + \\ &\quad + \exp(\mu(t-\tau))\Phi(t,\tau) \qquad\qquad \text{for} \quad \tau \leq t \\ &= \exp(\mu(T+t-\tau))\Phi(t,0)(I - \exp(\mu T)\Phi(T,0))^{-1}\Phi(T,\tau) \\ &\qquad\qquad \text{for} \quad \tau > t \ . \end{aligned}$$

The operator $g_\mu = \int_0^T G_\mu(t,\tau)\cdot d\tau$ in $L^2(J,H)$ is compact by Proposition 1.2.

Finally, if eqn. (4.3) has a weak, periodic solution $x(t)$, then by (4.4) and (4.6)

$$x(t) = -\int_0^T G_\mu(t,\tau)(C(\tau) + \mu I)x(\tau)d\tau$$

and if $x(t) \neq 0$, this leads to

$$\|x\|_{L^2} \leq \|g_\mu\| \cdot \sup_{t\in J} \|C(t)+\mu I\| \cdot \|x\|_{L^2} .$$

We assume now that α in (4.2) is less than $\|g_\mu\|^{-1}$. It then follows that

$$\|x\|_{L^2} \leq \|g_\mu\| \cdot \alpha \cdot \|x\|_{L^2} < \|x\|_{L^2}$$

which is a contradiction. Hence $x = 0$ is the only possible periodic solution of (4.3). We have therefore proved

Theorem 4.1. *Assume that* $B(t,x)$ *satisfies the Caratheodory condition described in* Theorem 3.1, *and that for all* $\rho(t) \in L^2(J,H)$, $B(t,\rho(t))$ *belongs to*

$$S = \{C(t) \in M(J,H) \,|\, \|\mu I + C(t)\| \leq \alpha < \|g_\mu\|^{-1}, \text{ for } \mu \in \mathbb{R},$$
$$\mu \neq -T^{-1}\log|\lambda_i| \text{ for all eigenvalues } \lambda_i \text{ of } \Phi(T,0)\} \qquad (4.7)$$

Further, let $A(t)$ *be as in* (4.1) *and let* $f(t,x)$ *satisfy the conditions in* Theorem 3.1. *Then the equation* (3.1) *has a weak, periodic solution.*

In Chapter III we shall give an example of an operator $B(t,x)$.

We will now describe a function $f(t,z)$ that conforms to the conditions of Theorem 3.1. Again let $H = L^2(\Omega)$.

Theorem 4.2. *Let* $0 < \gamma < 1$ *and* $h(t)$ *belong to the Lebesgue space* $L^\delta(J)$, *where* $\delta = 2/1-\gamma$. *If* $z \in H = L^2(\Omega)$, *then the function*

$$f(t,z) = h(t)|z|^\gamma \qquad (4.8)$$

satisfies the conditions of Theorem 3.1.

Proof. (1) $f(t,z)$ is measurable in t for each $z \in H$, since $h(t) \in L^{\delta}(J)$. (2) The following inequality holds

$$\|f(t,z)\|_H \leq C|h(t)|\cdot\|z\|_H^{\gamma} \quad . \tag{4.9}$$

In fact, since $p = 1/\gamma > 1$ we have for $y \in H$, by Hölder's inequality,

$$\int_\Omega (|y(x)|^{2\gamma})dx \leq \mu(\Omega)^{1-\gamma}(\int_\Omega (|y(x)|^{2\gamma})^{1/\gamma}dx)^{\gamma}$$

and therefore

$$\{\mu(\Omega)^{-1}\int_\Omega |y(x)|^{2\gamma}dx\}^{1/2\gamma} \leq \{\mu(\Omega)^{-1}\int_\Omega |y(x)|^2dx\}^{1/2}, \tag{4.10}$$

so that

$$\|f(t,z)\|_H^2 = |h(t)|^2\int_\Omega |z(x)|^{2\gamma}dx = |h(t)|^2\mu(\Omega)\{[\mu(\Omega)^{-1}\int_\Omega |z(x)|^{2\gamma}dx]^{1/2\gamma}\}^{2\gamma}$$

$$\leq |h(t)|^2\mu(\Omega)\{\mu(\Omega)^{-1/2}\|z\|_H\}^{2\gamma} = |h(t)|^2\mu(\Omega)^{1-\gamma}\|z\|_H^{2\gamma}.$$

(3) Let $z \in L^2(J,H)$. Using (4.9) we have

$$\int_0^T \|f(t,z(t))\|_H^2dt \leq C^2\int_0^T |h(t)|^2\cdot\|z(t)\|_H^{2\gamma}dt$$

$$\leq C^2\{\int_0^T |h(t)|^{2/1-\gamma}dt\}^{1-\gamma}\{\int_0^T \|z(t)\|_H^2dt\}^{\gamma} < \infty,$$

by applying Hölder's inequality with $p = 1/1-\gamma$. Therefore $f(t,z(t))$ belongs to $L^2(J,H)$.

(4) Let (z_ℓ) be a sequence in $L^2(J,H)$ with $\|z_\ell\| \leq \ell$. Then by (4.9)

$$Q_\ell = \ell^{-1}\int_0^T \|f(t,z_\ell(t))\|_H dt \leq C\ell^{-1}\int_0^T |h(t)|\cdot\|z_\ell(t)\|^{\gamma}dt$$

$$\leq C\ell^{-1}\{\int_0^T |h(t)|^{1/1-\gamma}dt\}^{1-\gamma}\{\int_0^T \|z_\ell(t)\|dt\}^{\gamma}, \tag{4.11}$$

again using Hölder, $p = 1/1-\gamma$. By the assumption on $h(t)$ and Hölder's inequality it follows that the first integral in the product in (4.11) is finite. Also

$$\left\{\int_0^T \|z_\ell(t)\|dt\right\}^\gamma \le \{T^{1/2}\|z_\ell\|_{L^2(J,H)}\}^\gamma \le T^{\gamma/2}\ell^\gamma .$$

Hence in (4.11) $Q_\ell \le P\ell^{\gamma-1} \to 0$ as $\ell \to \infty$, where P is a constant.

(5) We show that $f(t,z)$ is continuous in z for almost every $t \in J$. Let $z_n \to y$ in H. By virtue of (4.10) $(|z_n|^\gamma)$ is a bounded sequence in H. We now show that

$$\|f(t,z_n) - f(t,y)\|_H^2 = |h(t)|^2\int_\Omega \left||z_n(x)|^\gamma - |y(x)|^\gamma\right|^2 dx \to 0$$

as $n \to \infty$. Since $\|z_n - y\|_H \to 0$, we can find a subsequence (j) such that $\left||z_j(x)| - |y(x)|\right| \le |z_j(x) - y(x)| \to 0$ as $j \to \infty$, for almost every $x \in \Omega$. Hence for such x, $|z_j(x)|^\gamma \to |y(x)|^\gamma$. Since $z_n \to y$ in $L^2(\Omega)$ and in view of [17; Vol.I,p122, theorem 6] and inequality (4.10) $(0 < \gamma < 1)$, the conditions of Vitali's convergence theorem [17; Vol.I,p150] are satisfied. It therefore follows that $|z_j|^\gamma \to |y|^\gamma$ in $L^2(\Omega)$ as $j \to \infty$.

Since every subsequence (z_i) of (z_n) converges to y, every subsequence $(|z_i|^\gamma)$ will, by the same argument, contain a subsequence $(|z_\ell|^\gamma)$ which will converge to $|y|^\gamma$. Hence the whole sequence $(|z_n|^\gamma)$ converges to $|y|^\gamma$, i.e. $f(t,z)$ is continuous. Thus the sublinear function as given in (4.8) satisfies the conditions of Theorem 3.1. With this we conclude this section □.

Remark. Functions $f(t,z)$ which satisfy an inequality of type (4.9) are used for example in [11; p141-159] and Conti [13; p158], note also Martin [51; p161].

CHAPTER II

0. INTRODUCTION

This chapter will be concerned with the semilinear evolution equation and the quasi-periodic boundary condition

$$x' + (A(t) + B(t,\zeta(x)))x = f(t,x) \tag{0.1}$$

$$kx(0) = x(T) \quad k \in \mathbb{R},$$

in a separable Hilbert space H.

As in Chapter I, $A(t)$ is a function from $J = [0,T]$ to the set of unbounded operators acting in H, but here satisfying all four of the assumptions of Kato, Tanabe [41;(E.1)-(E.4)]. Again $A(t)$ must generate a compact semigroup for every $t \in J$. The function $f(t,x)$ is the same as in Theorem 3.1, Chapter I. In contrast to Chapter I, the perturbation operator $B(t)$ may now be unbounded, but subordinate to $A(t)$ in the sense of [41;(E.5)-(E.6)]. The purpose of the function ζ in $B(t,\zeta(\rho))$ is to reconcile the fact that $\rho \in L^2(J,H)$ with the conditions (E.5), (E.6) of [41] which require that $B(t)$ be defined for all $t \in J$, see (1.10).

Instead of proving the existence of a periodic solution of equation (0.1), we will only be able to assert in Theorem 2.1 that a quasi-periodic solution exists, for sufficiently large k. This is the content of Section 2. In the first section we will introduce an unbounded nonlinear perturbation operator $B(t,\zeta(\rho))$ and prove a convergence result (Theorem 1.7). We do not need a priori estimates on first derivatives. This result will be instrumental in the proof of the main theorem (Theorem 2.1).

The 3rd section contains an example illustrating the results obtained in Section 2. It is based on the Sturm-Liouville boundary value problem and has time dependent boundary conditions. If these change smoothly, then so do the corresponding eigenvalues $\lambda_n(t)$. Moreover we will show that $(\lambda_n(t))^{-1}d\lambda_n/dt$ is uniformly bounded in n and $t \in J$.

We mention here that the continuity in the uniform operator topology of the evolution operator Ψ of (1.1), (Theorem 1.3) is not needed in the proof of the main result. We have included the result because it is of interest in itself.

1. PERTURBED EQUATIONS OF EVOLUTION

This section will be concerned with the abstract evolution equation

$$x' + (A(t) + B(t))x = 0,\ 0 \le t \le T,\ ('=d/dt), \tag{1.1}$$

in a separable Hilbert space H. $A(t)$ is a function from $J = [0,T]$ to the set of unbounded operators acting in H. It is assumed to satisfy the conditions (E.1) - (E.4) of Kato, Tanabe [41; p109,110]. Again for every $t \in J$, $A(t)$ generates a compact semigroup.

The perturbation operator $B(t)$ is unbounded, but subordinate to $A(t)$ in the sense of (AII.2), (AII.3) below.

We will first introduce all our assumptions. Since we base our results mainly on Kato and Tanabe's paper [41, Section 6], we compile some of their results which we shall need, in a separate paragraph.

The evolution operator $\Psi(t,s)$ of (1.1) will then be shown to be continuous in the uniform operator topology, and because of the compactness assumption it will once again follow that $\Psi(t,s)$ is compact for $t > s$.

In the second part of the section, we will introduce the nonlinear perturbation operator $B(t,\zeta(\rho))$. If the bounded operator $B(t,x)(\lambda I+A(t))^{-1}$ in H is strongly continuous in x, $x \in H$, then it will follow that if $\rho_n \to \rho$ in $L^2(J,H)$, the sequence of evolution operators $\Psi_n(t,s)$ of $x' + (A(t) + B(t,\zeta(\rho_n)))x = 0$ is strongly convergent in H for $t > s$. This will be proved in Theorem 1.7. With this result available, it will be possible in the succeeding section to prove that equation (0.1) has a quasi-periodic solution. We will denote with $\Phi_0(t,s)$ the evolution operator of

$$x' + A(t)x = 0 \tag{1.2}$$

Subsequently assumptions (E.1) - (E.6) of [41] are made, i.e. we propose that the operator $A(t)$ satisfies the conditions (AI.1) - (AI.4) of Chapter I as well as

(AII.1) $dA(t)^{-1}/dt$ is Hölder continuous with respect to $t \in J$ in the uniform operator topology, i.e. there exist positive constants K_1 and α such that for each $t,s \in J$

$$\| dA(t)^{-1}/dt - dA(s)^{-1}/ds \| \le K_1 |t-s|^{\alpha}.$$

Concerning $B(t)$ we assume that

(AII.2) For each $t \in J$, $B(t)$ is a closed linear operator whose domain contains that of $A(t)$. There exist positive constants M_1 and γ, $0 < \gamma < 1/2$, such that for each $\lambda \in \Sigma$ (Σ as in (AI.1)) and $t \in J$,

$$\| B(t)(\lambda I + A(t))^{-1} \| \le M_1/|\lambda|^{1-\gamma} .$$

(AII.3) There exist positive constants K_2 and β such that for each $t,s \in J$,

$$\| B(t)A(t)^{-1} - B(s)A(s)^{-1} \| \le K_2 |t-s|^{\beta} .$$

Remark. 1) As we are using the notion of a strict solution [41;p111] in this chapter, the extra assumption (AII.1) on $A(t)$ is therefore necessary. Also for example, Lemma 1.6 requires (AII.1), where we need $\Psi(t,s)x$ to be in the domain of $A(t)$ for every $x \in H$ and $s < t$.
2) In (AII.2) our assumption on γ is more restrictive than in [41; (E.5), $\gamma<1$]. We require it though, since we work in $L^2(J,H)$. It remains an open question as to whether this chapter could be dealt with in a reflexive Banach space setting. Thereby the restriction on γ could be relaxed.

In this paragraph we shall compile some of the results given in [41; p122-125] which will be used in the sequel. By (AII.2) one obtains for $t > 0$

$$\| B(s)\exp(-tA(s))\| \leq C\, t^{-\gamma} \tag{1.3}$$

and hence

$$\| B(t)\Phi_0(t,s)\| \leq M_2(t-s)^{-\gamma}, \tag{1.4}$$

where $\exp(-tA(s))$ denotes the analytic semigroup generated by $-A(s)$, and $\Phi_0(t,s)$ is the evolution operator of (1.2).

The evolution operator $\Psi(t,s)$ of (1.1) is formally constructed by

$$\Psi(t,s) = \sum_{m=0}^{\infty} (-1)^m \Phi_m(t,s) \tag{1.5}$$

where $\Phi_m(t,s) = \int_s^t \Phi_0(t,\sigma)B(\sigma)\Phi_{m-1}(\sigma,s)d\sigma \quad m = 1,2,\ldots .$

If M is the constant in (C.1) (Chapter I), i.e. $\|\Phi_0(t,s)\| \leq M$, then by induction one obtains

$$\begin{aligned} &\| \Phi_m(t,s)\| \leq M\, M_2^m \Gamma(1-\gamma)^m (t-s)^{m(1-\gamma)}/\Gamma((m+1)(1-\gamma)), \\ &\text{and} \\ &\| B(t)\Phi_m(t,s)\| \leq M_2^{m+1}\Gamma(1-\gamma)^{m+1}(t-s)^{m(1-\gamma)-\gamma}/\Gamma((m+1)(1-\gamma)). \end{aligned} \tag{1.6}$$

It follows that

$$\begin{aligned} \Psi(t,s) &= \Phi_0(t,s) - \int_s^t \Phi_0(t,\sigma)B(\sigma)\Psi(\sigma,s)d\sigma \\ &= \Phi_0(t,s) - \int_s^t \Psi(t,\sigma)B(\sigma)\Phi_0(\sigma,s)d\sigma. \end{aligned} \tag{1.7}$$

The notion of a strict solution is introduced now, [41; p111].

Definition 1.1. The function $u(t)$ is called a *strict solution* of

$$x' + (A(t) + B(t))x = f(t) \tag{1.8}$$

in $(s,T]$, where $f(t) \in C(J,H)$, if

(1) $u(t)$ is strongly continuous in $[s,T]$ and strongly continuously differentiable in the interval $(s,T]$;

(2) for each $t \in (s,T]$, $u(t)$ belongs to $\mathcal{D}(A(t))$, the domain of $A(t)$.

(3) $u(t)$ satisfies (1.8) in $(s,T]$.

The main results in [41; p122-125] are summarized in

Theorem 1.1. *Under the assumptions* (AI.1) - (AI.3) *and* (AII.1) - (AII.3) *an evolution operator* $\Psi(t,s)$ *exists for the perturbed equation* (1.1), *and it satisfies*

$$\|(\partial/\partial t)\Psi(t,s)\| \leq C/(t-s), \quad \|A(t)\Psi(t,s)\| \leq C/(t-s)$$

$$\|B(t)\Psi(t,s)\| \leq C/(t-s)^{\gamma}$$

$$\frac{\partial}{\partial s}\Psi(t,s)u = \Psi(t,s)(A(s)+B(s))u, \qquad \text{for} \quad u \in \mathcal{D}(A(s))$$

$$\Psi(t,r)\Psi(r,s) = \Psi(t,s), \qquad s \leq r \leq t .$$

Let $f(t)$ *be Hölder continuous in* $(s,T]$. *Then the unique strict solution of* (1.8) *in* $(s,T]$ *is given by*

$$u(t) = \Psi(t,s)u(s) + \int_s^t \Psi(t,\sigma)f(\sigma)d\sigma.$$

Remark. There is a sign confusion in [41; p122-125 (e.g. (6.6),(6.14), (6.18))]. This can be remedied by constructing the evolution operator as in (1.5) - slightly differently to [41; (6.6),(6.7)].

We establish now the compactness of the evolution operator $\Psi(t,s)$ of (1.1) for $s < t$. It will follow from that of $\Phi_0(t,s)$ and the integral representation for $\Psi(t,s)$ given in (1.7).

Theorem 1.2. *If* $A(t)$ *and* $B(t)$ *satisfy the assumptions* (AI.1) - (AI.4) *and* (AII.1) - (AII.3), *then the evolution operator* $\Psi(t,s)$ *of eqn.* (1.1) *is compact for* $s < t$.

Proof. By (1.7)

$$\Psi(t,s) = \Phi_0(t,s) - \int_s^t \Phi_0(t,\sigma)B(\sigma)\Psi(\sigma,s)d\sigma.$$

The evolution operator $\Phi_0(t,s)$ of (1.2) is compact for $s < t$, by I Theorem 2.1, and by Theorem 1.1, $B(\sigma)\Psi(\sigma,s)$ is a bounded, linear operator for almost every $\sigma \in (s,t)$. Therefore $\Phi_0(t,\sigma)B(\sigma)\Psi(\sigma,s)$ is compact. Thus for a sequence (x_n) weakly converging to x in H,

$$\Phi_0(t,\sigma)B(\sigma)\Psi(\sigma,s)x_n \to \Phi_0(t,\sigma)B(\sigma)\Psi(\sigma,s)x \quad (n\to\infty),$$

strongly for almost every $\sigma \in (s,t)$. Since (x_n) is a bounded sequence the dominated convergence theorem guarantees that $\Psi(t,s)x_n \to \Psi(t,s)x$ $(n\to\infty)$. Hence $\Psi(t,s)$ is a compact operator for $s < t$ □.

The next theorem concerns the continuity of $\Psi(t,s)$ in the uniform operator topology. The proof is similar to that of I Theorem 2.2.

Theorem 1.3.* *Let* $A(t)$ *and* $B(t)$ *satisfy the assumptions* (AI.1) - (AI.3) *and* (AII.1) - (AII.3). *Then the evolution operator* $\Psi(t,s)$ *of* (1.1) *is continuous in the uniform operator topology in*

$$S' = \{(t,s) \in J^2 | s < t\}.$$

Proof. We will use the two forms of $\Psi(t,s)$ given in (1.7) and the continuity in S' of $\Phi_0(t,s)$ of eqn. (1.2) as guaranteed by I Theorem 2.2. (As we have remarked after the proof of I Theorem 2.2, assumption (AI.4) is not required.)

We verify the continuity separately for t and s.

---

*As remarked earlier, this result is not needed for the proof of the main theorem.

(i) *Continuity in* t

Let $s < t_1 < t_2$. Then by (1.7)

$$\Psi(t_1,s) - \Psi(t_2,s) = \Phi_0(t_1,s) - \Phi_0(t_2,s) + \int_{t_1}^{t_2} \Phi_0(t_2,\sigma)B(\sigma)\Psi(\sigma,s)d\sigma$$

$$+ \int_s^{t_1} [\Phi_0(t_2,\sigma) - \Phi_0(t_1,\sigma)]B(\sigma)\Psi(\sigma,s)d\sigma.$$

The norms of the first two expressions on the right-hand side in the above equality clearly converge to zero as $|t_1 - t_2| \to 0$.

Concerning the third expression, let t_1 be fixed first. By Theorem 1.1, the norm of its integrand is bounded by $2MC(\sigma-s)^{-\gamma}$, which is integrable. Further, it converges to zero for almost every $\sigma \in (s,t_1)$. Therefore the result follows by an application of the dominated convergence theorem.

If t_2 is fixed, we define the function

$$F(t_1,\sigma) = \|[\Phi_0(t_2,\sigma) - \Phi_0(t_1,\sigma)]B(\sigma)\Psi(\sigma,s)\|, \quad \text{for} \quad s \le \sigma \le t_1$$

$$= \quad 0 \qquad \text{for} \quad t_1 < \sigma \le t_2,$$

so that

$$\int_s^{t_1} \|[\Phi_0(t_2,\sigma) - \Phi_0(t_1,\sigma)]B(\sigma)\Psi(\sigma,s)\| d\sigma = \int_s^{t_2} F(t_1,\sigma)d\sigma.$$

The function $F(t_1,\sigma)$ is bounded by $2MC(\sigma-s)^{-\gamma}$ which is integrable. In order to apply the dominated convergence theorem, it remains to show that

$$\lim_{t_1 \to t_2} F(t_1,\sigma) = 0 \quad \text{for almost every} \quad \sigma \in (s,t_2).$$

Since σ remains fixed while t_1 approaches t_2, we may assume that $\sigma < t_1 < t_2$ for almost every $\sigma \in (s,t_2)$. Therefore

$$F(t_1,\sigma) = \| [\Phi_0(t_2,\sigma) - \Phi_0(t_1,\sigma)] B(\sigma)\Psi(\sigma,s)\|$$

and hence does approach zero as $t_1 \to t_2$, for almost every $\sigma \in (s,t_2)$. This completes the proof that $\Psi(t,s)$ is continuous in t. (See also [41; (6.16)]).

(ii) *Continuity in* s

Let $s_2 < s_1 < t$. Then by (1.7)

$$\Psi(t,s_1) - \Psi(t,s_2) = \Phi_0(t,s_1) - \Phi_0(t,s_2) + \int_{s_2}^{s_1} \Psi(t,\sigma)B(\sigma)\Phi_0(\sigma,s_2)d\sigma$$

$$+ \int_{s_1}^{t} \Psi(t,\sigma)B(\sigma)[\Phi_0(\sigma,s_2) - \Phi_0(\sigma,s_1)]d\sigma$$

The first two expressions on the right-hand side in the above equation converge to zero in norm, as $|s_1-s_2| \to 0$, because of I Theorem 2.2, and (1.4). We now show that

$$\int_{s_1}^{t} \Psi(t,\sigma)B(\sigma)[\Phi_0(\sigma,s_2) - \Phi_0(\sigma,s_1)]d\sigma$$

approaches zero in norm, as $|s_1-s_2| \to 0$. Let s_1 be fixed first. Thus by I Definition 2.1, the integrand can be written as

$$\Psi(t,\sigma)B(\sigma)\Phi_0(\sigma,s_1)[\Phi_0(s_1,s_2) - I]$$

which is seen to be bounded by $KM_2(M+1)(\sigma-s_1)^{-\gamma}$, setting $\|\Psi(t,\sigma)\| \le K$. This expression is integrable.

Let $\sigma \in (s_1,t)$, then there exists τ with $s_1 < \tau < \sigma$. Hence the integrand can be written in the form

$$\Psi(t,\sigma)B(\sigma)\Phi_0(\sigma,\tau)[\Phi_0(\tau,s_2) - \Phi_0(\tau,s_1)],$$

the norm of which converges to zero as $s_2 \to s_1$. Therefore the result follows with the dominated convergence theorem.

Now let s_2 be fixed. Thus $t - s_2$ is a given positive number. To a given $\varepsilon > 0$, we choose $\kappa > 0$ such that $2KM_2\kappa^{1-\gamma}/(1-\gamma) < \varepsilon/2$. Because of the continuity of $\Phi_0(t,s)$ in S', we can find δ, $0 < \delta < (t-s_2)/3$ such that $|s_1-s_2| < \delta$ implies

$$\|\Phi_0(s_1+\kappa,s_2) - \Phi_0(s_1+\kappa,s_1)\| < \varepsilon/2Q$$

where $Q = KM_2(t-s_2)^{1-\gamma}/(1-\gamma)$. Hence

$$\int_{s_1}^{t} \|\Psi(t,\sigma)B(\sigma)[\Phi_0(\sigma,s_2) - \Phi_0(\sigma,s_1)]\|d\sigma$$

$$\leq \int_{s_1}^{s_1+\kappa} K(\|B(\sigma)\Phi_0(\sigma,s_2)\| + \|B(\sigma)\Phi_0(\sigma,s_1)\|)d\sigma$$

$$+ \int_{s_1+\kappa}^{t} K\|B(\sigma)\Phi_0(\sigma,s_1+\kappa)\|\|\Phi_0(s_1+\kappa,s_2) - \Phi_0(s_1+\kappa,s_1)\|d\sigma$$

By (1.4) the first integral on the right-hand side of the above inequality is less than or equal to

$$KM_2\int_{s_1}^{s_1+\kappa} (\sigma-s_2)^{-\gamma} + (\sigma-s_1)^{-\gamma}d\sigma = KM_2\{(s_1-s_2+\kappa)^{1-\gamma} - (s_1-s_2)^{1-\gamma} + \kappa^{1-\gamma}\}/(1-\gamma)$$

$$\leq 2KM_2\kappa^{1-\gamma}/(1-\gamma) < \varepsilon/2 ,$$

by similar methods to those used in the proof of I Theorem 2.2. The second integral on the right-hand side of the above inequality is less than

$$(\varepsilon KM_2/2Q)\int_{s_1+\kappa}^{t} (\sigma-s_1-\kappa)^{-\gamma}d\sigma \leq \varepsilon KM_2(t-s_2)^{1-\gamma}/2(1-\gamma)Q < \varepsilon/2 .$$

We have thereby shown that $\Psi(t,s)$ is continuous in s, $s < t$, and the proof is complete $\square$.

In this paragraph we will describe a non-linear perturbation operator B and investigate the convergence properties of the corresponding evolution operator of eqn (1.1). The main result is contained in Theorem 1.7.

Let B' be a map from J into the set of closed linear operators in H. We assume that $B'(t)$ satisfies (AII.2) and (AII.3). By fixing the constants in (AII.2) we form the set

$$N = \{B(t) \mid B(t) \text{ satisfies (AII.2) for given fixed constants } M_1 \text{ and } \gamma \text{, and it satisfies (AII.3)}\} . \tag{1.9}$$

With the help of (1.5) and (1.6) we see that the evolution operator $\Psi(t,s)$ of (1.1) is uniformly bounded for $B'(t) \in N$.

Now let B denote a map from $J \times H$ into the set of closed linear operators in H and further let ζ be the function from $L^2(J,H)$ into the space of the continuous functions $C(J,H)$, given by

$$\zeta(\rho)(t) = \int_0^t \rho(\tau)d\tau + c \tag{1.10}$$

for some constant $c \in H$. (One could choose a different subinterval of J in the definition of ζ or one could define ζ to be the mollifier for a given positive ε). We *define* the non-linear operator B on $L^2(J,H)$ by

$$B(\rho) = B(t,\zeta(\rho)(t)) \tag{1.11}$$

and assume that it has range in N.

Remark. As we mentioned before, the properties in (AII.2), (AII.3) are assumed to hold for all $t \in J$. Since our function space is the

Hilbert space $L^2(J,H)$ a function ζ of type (1.10) had to be introduced.

In the sequel the abbreviation $B_\rho(t) = B(t,\zeta(\rho)(t))$ will be used and further, $\Psi_\rho(t,s)$ will denote the evolution operator of

$$x' + (A(t) + B_\rho(t))x = 0$$

Since we assume that $B_\rho(t)$ belongs to N for every $\rho \in L^2(J,H)$, we obtain by (1.5) and (1.6)

$$\|\Psi_\rho(t,s)\| \leq K , \tag{1.12}$$

where K is independent of ρ , s and t .

We will require the following three lemmas in the proof of the convergence result (Theorem 1.7).

Lemma 1.4. *Assume that* $B(t,\zeta(\rho)(t)) \in N$ *for* $\rho \in L^2(J,H)$, *and further that for every* $t \in J$ *and* $\lambda \in \Sigma$, *the bounded linear operator of* (AII.2)

$$B(t,x)(\lambda I+A(t))^{-1}$$

is strongly continuous in x, $x \in H$. *Then for every* $s, t \in J$ *with* $s < t$ *and* $x_0 \in H$, *the non-linear operator*

$$B(\cdot)\exp(-(t-s)A(t))x_0 : L^2(J,H) \to H$$

is continuous, where $\exp(-\tau A(t))$ *denotes the semigroup generated by* $A(t)$.

Proof. Since $\exp(-\tau A(t))$ is an analytic semigroup, $\tau > 0$, and $B_\rho(t) \in N$, the above expressions are well defined.

Letting $\rho_n \to \rho$ in $L^2(J,H)$ $(n\to\infty)$, it follows that $\zeta(\rho_n)(t) \to \zeta(\rho)(t)$

in H, uniformly in t. Therefore by assumption

$$[B(t,\zeta(\rho_n)(t)) - B(t,\zeta(\rho)(t))](\lambda I+A(t))^{-1}x_0$$

converges to zero in norm as $n \to \infty$, for any $t \in J$, $\lambda \in \Sigma$ and $x_0 \in H$. Using [41;(1.2)] and [17; Vol.I,p153] we obtain

$$[B(t,\zeta(\rho_n)(t)) - B(t,\zeta(\rho)(t))]\exp(-(t-s)A(t))x_0$$

$$= (2\pi i)^{-1}\int_\Gamma e^{\lambda(t-s)}[B(t,\zeta(\rho_n)(t)) - B(t,\zeta(\rho)(t))](\lambda I+A(t))^{-1}x_0\, d\lambda,$$

where Γ is a smooth contour running in Σ from $\infty e^{-i(\pi/2+\theta)}$ to $\infty e^{i(\pi/2+\theta)}$, $(0<\theta<\pi/2)$.

The norm of the integrand in the above integral converges to zero $(n\to\infty)$ for every $\lambda \in \Gamma$ and any s and t, $s < t$, and $x_0 \in H$. Moreover the integrand is bounded by $2|e^{\lambda(t-s)}|\|x_0\|M_1|\lambda|^{\gamma-1}$ which is integrable. The conclusion now follows with the dominated convergence theorem □.

Lemma 1.5. *Under the hypothesis of* Lemma 1.4, *the non-linear operator*

$$B(\cdot)\Phi_0(t,s)x_0 \;:\; L^2(J,H) \to H$$

is continuous for every $s, t \in J$ *with* $s < t$, *and* $x_0 \in H$. (Φ_0 *denotes the evolution operator of* (1.2).)

Proof. We use the previous lemma, I(2.4), and [17; Vol.I,p153]. Accordingly, let $\rho_n \to \rho$ in $L^2(J,H)$, then, setting $B_n(t) = B(t,\zeta(\rho_n)(t))$ we obtain

$$[B(\rho_n) - B(\rho)]\Phi_0(t,s)x_0 = [B_n(t) - B_\rho(t)]\exp(-(t-s)A(t))x_0$$
$$+ \int_s^t [B_n(t) - B_\rho(t)]\exp(-(t-\sigma)A(t))R(\sigma,s)x_0\, d\sigma.$$

By Lemma 1.4, we only need to consider the integral in the above equation. Its integrand will approach zero for almost every $\sigma \in (s,t)$, as $n \to \infty$, by Lemma 1.4. Furthermore by assumption, the proof of the previous lemma, (1.3) and I(2.5), we see that the integrand is bounded by $2C\|x_0\|(t-\sigma)^{-\gamma}(\sigma-s)^{-\rho}$ which is integrable, since it reduces to a Beta-function. Hence the result follows by another application of the dominated convergence theorem □

The following lemma is a useful technical tool. Since we require in the proof, that the range of $\Psi(t,s)$, $s < t$, is contained in $\mathcal{D}(A(t))$, assumption (AII.1) is unavoidable.

Lemma 1.6. *Let* $B_1(t)$, $B_2(t) \in N$, *and denote with* $\Psi_i(t,s)(i=1,2)$ *the corresponding evolution operators of equation* (1.1). *Then for each* $x \in H$ *and* $s < t$,

$$\Psi_1(t,s)x - \Psi_2(t,s)x = \int_s^t \Psi_2(t,\sigma)[B_2(\sigma) - B_1(\sigma)]\Psi_1(\sigma,s)x\,d\sigma .$$

Proof. Using [41; (6.18),(6.19)], see also Theorem 1.1, one obtains for $s < \sigma < t$,

$$\frac{\partial}{\partial\sigma}\{\Psi_2(t,\sigma)\Psi_1(\sigma,s)x\} = \frac{\partial}{\partial\sigma}\Psi_2(t,\sigma)(\Psi_1(\sigma,s)x) + \Psi_2(t,\sigma)\frac{\partial}{\partial\sigma}\Psi_1(\sigma,s)x$$

$$= \Psi_2(t,\sigma)[A(\sigma) + B_2(\sigma)]\Psi_1(\sigma,s)x - \Psi_2(t,\sigma)[A(\sigma) + B_1(\sigma)]\Psi_1(\sigma,s)x$$

$$= \Psi_2(t,\sigma)[B_2(\sigma) - B_1(\sigma)]\Psi_1(\sigma,s)x .$$

Upon integration, the result follows □ .

By virtue of the following convergence result and I Proposition 2.9, we will be able to prove the continuity of the operator G in Theorem 2.1.

<u>Theorem 1.7</u>. *Assume that for* $\rho \in L^2(J,H)$, $B(t,\zeta(\rho)(t))$ *of* (1.11) *belongs to the set* N *of* (1.9). *Further assume that for every* $t \in J$ *and* $\lambda \in \Sigma$ *the bounded linear operator in* H

$$B(t,x)(\lambda I+A(t))^{-1}$$

is strongly continuous in x, $x \in H$. *Then if* $\rho_n \to \rho$ $(n\to\infty)$ *in* $L^2(J,H)$, *the corresponding evolution operators* $\Psi_n(t,s)$ *of*

$$x' + (A(t) + B(t,\zeta(\rho_n)(t)))x = 0$$

converge strongly in H *to* $\Psi_\rho(t,s)$ *for* $s < t$.

<u>Proof</u>. Let $s < t$, $x_0 \in H$ and suppose that $\rho_n \to \rho$ in $L^2(J,H)$, $(n\to\infty)$. We abbreviate $B(t,\zeta(\rho_n)(t))$ to $B_n(t)$ and as before $B(t,\zeta(\rho)(t))$ to $B_\rho(t)$. Using Lemma 1.6, (1.7), (1.12) and [17; Vol.I,p153] one obtains

$$\|\Psi_\rho(t,s)x_0 - \Psi_n(t,s)x_0\| \le K\int_s^t \|[B_n(\sigma) - B_\rho(\sigma)]\Psi_\rho(\sigma,s)x_0\| d\sigma$$

$$\le K\{\int_s^t \|[B_n(\sigma) - B_\rho(\sigma)]\Phi_0(\sigma,s)x_0\| d\sigma +$$

$$\int_s^t\int_s^\sigma \|[B_n(\sigma) - B_\rho(\sigma)]\Phi_0(\sigma,\mu)B_\rho(\mu)\Psi_\rho(\mu,s)x_0\| d\mu d\sigma\}$$

The integrand in the first integral on the right-hand side of the last inequality approaches zero $(n\to\infty)$ for almost every $\sigma \in (s,t)$, by Lemma 1.5. Further by assumption and (1.4) it is bounded by $2\|x_0\|M_2(\sigma-s)^{-\gamma}$ which is integrable. Thus dominated convergence guarantees that this integral converges to zero as $n \to \infty$.

The integrand in the second integral on the right-hand side of the last inequality converges to zero $(n\to\infty)$ by Lemma 1.5 and Theorem 1.1, for almost every $\sigma \in (s,t)$ and $\mu \in (s,\sigma)$. Also it is bounded by

$2M_2C\|x_0\|(\sigma-\mu)^{-\gamma}(\mu-s)^{-\gamma}$ as can be seen from (1.4) and Theorem 1.1. With the two transformations $u = \sigma-\mu$ and $v(\sigma-s) = u$ we see that the integral

$$\int_s^t\int_s^\sigma (\sigma-\mu)^{-\gamma}(\mu-s)^{-\gamma}d\mu d\sigma = \int_s^t (\sigma-s)^{1-2\gamma}\int_0^1 v^{-\gamma}(1-v)^{-\gamma}dvd\sigma$$

$$= (t-s)^{2-2\gamma}\Gamma(1-\gamma)^2/(2-2\gamma)\Gamma(2-2\gamma)$$

exists. Therefore by yet another application of the dominated convergence theorem, the result follows □ .

2. QUASI-PERIODIC SOLUTIONS

In this section we shall prove that the semilinear equation of evolution

$$x' + (A(t) + B(t,\zeta(x)))x = f(t,x) \tag{2.1}$$

has a quasi-periodic solution. This will be achieved by an application of Schauder's fixed point theorem. The perturbation $B(t,\zeta(x))$ is given by (1.11), and $f(t,x)$ satisfies the conditions in Theorem 2.1, and Corollary 2.2.

In Theorem 2.1 it will first be shown that equation (2.1) has a mild, quasi-periodic solution (Definition 2.1). With the additional Hölder continuity assumption in Corollary 2.2, equation (2.1) has a strict, quasi-periodic solution. This type of boundary condition is described for example in Conti [13;p146,163].

Definition 2.1. A continuous function $y(t)$ on $J = [0,T]$ that satisfies

$$x(t) = \Phi_0(t,0)x_0 - \int_0^t \Phi_0(t,\sigma)[B(\sigma,\zeta(x)(\sigma))x(\sigma) - f(\sigma,x(\sigma))]d\sigma \tag{2.2}$$

and $ky(0) = y(T)$ for some $k \in \mathbb{R}$

is called a *mild, quasi-periodic solution* of equation (2.1). ($\Phi_0(t,s)$ denotes the evolution operator of (1.2).)

Theorem 2.1. *Let the following be valid.* $A(t)$ *satisfies* (AI.1) - (AI.4) *of Chapter I and* (AII.1).
For every $\rho(t) \in L^2(J,H)$, $B(t,\zeta(\rho)(t))$ *of* (1.11) *belongs to the set* N *of* (1.9), *and further for every* $t \in J$ *and* $\lambda \in \Sigma$, *the bounded*

linear operator in H

$$B(t,x)(\lambda I+A(t))^{-1}$$

is assumed to be strongly continuous in x, $x \in H$.

Concerning $f(t,x): J \times H \to H$ *we assume that*

(1) *it is measurable in* t , *for each* $x \in H$, *and continuous in* x , *for almost all* $t \in J$.

(2) *For every* $\rho(t) \in L^2(J,H)$, $f(t,\rho(t)) \in L^2(J,H)$.

(3) *For every sequence* (ρ_n) *in* $L^2(J,H)$, *with* $\|\rho_n\| \le n$

$$\liminf_{n \to \infty}(1/n)\int_0^T \|f(t,\rho_n(t))\|\,dt = 0 \tag{2.3}$$

Then equation (2.1) *has a mild, quasi-periodic solution for any* $k \in \mathbb{R}$ *with* $|k| > L$, *where by* (1.12) $L = \sup_{\rho \in L^2(J,H)} \overline{\lim_n} \|\Psi_\rho(T,0)^n\|^{1/n} \le K$.

Remark. The uniqueness assumption of I Theorem 3.1 is replaced here by the weaker result of a quasi-periodic solution with the condition $|k| > L$.

In Chapter I, the uniqueness assumption of I Theorem 3.1, together with I Proposition 1.1 guaranteed a uniform bound for $(I-\Psi_\rho(T,0))^{-1}$. Here we obtain a similar bound straight from the assumptions. Because of it a priori estimates of the derivative are again not needed.

Is it possible to find a uniform bound for $(I-\Psi_\rho(T,0))^{-1}$, $\rho \in L^2(J,H)$, assuming uniqueness as in I Theorem 3.1, in the case of unbounded perturbations? If so, eqn. (2.1) would have a periodic solution. The question seems to be open.

Proof. Given $\rho \in L^2(J,H)$, we will abbreviate $B(t,\zeta(\rho)(t))$ to $B_\rho(t)$ and $f(t,\rho(t))$ to $f_\rho(t)$. By Theorem 1.1, the evolution operator of

$$x' + (A(t) + B_\rho(t))x = f_\rho(t) \tag{2.4}$$

exists and is unique. We will denote it by $\Psi_\rho(t,s)$.

Let $k \in \mathbb{R}$ satisfy $|k| > L$ as in the statement. Since

$$(kI-\Psi_\rho(T,0))^{-1} = \sum_{n=0}^{\infty} \Psi_\rho(T,0)^n/k^{n+1} ,$$

referring to [3; p82], we see that

$$\|(kI-\Psi_\rho(T,0))^{-1}\| \le (|k|-L)^{-1} = P , \tag{2.5}$$

for any $\rho \in L^2(J,H)$. It follows now that for any $\rho \in L^2(J,H)$, the function

$$\psi_\rho(t) = \Psi_\rho(t,0)x_\rho + \int_0^t \Psi_\rho(t,\sigma)f_\rho(\sigma)d\sigma \tag{2.6}$$

with

$$k\psi_\rho(0) = \psi_\rho(T)$$

is uniquely determined, namely

$$x_\rho = (kI-\Psi_\rho(T,0))^{-1}\int_0^T \Psi_\rho(T,\sigma)f_\rho(\sigma)d\sigma . \tag{2.7}$$

It is easy to see that $\psi_\rho(t)$ is continuous. We note that if $f_\rho(\sigma)$ is Hōldercontinuous, then by Theorem 1.1, the function in (2.6) is the unique strict solution of (2.4). For any $\rho \in L^2(J,H)$, $\psi_\rho(t)$ of (2.6) satisfies

$$x(t) = \Phi_0(t,0)x_\rho - \int_0^t \Phi_0(t,\sigma)[B_\rho(\sigma)x(\sigma)-f_\rho(\sigma)]d\sigma . \tag{2.8}$$

This can be seen by using (1.7) and Fubini's theorem in (2.6).

In fact

$$\psi_\rho(t) = \Phi_0(t,0)x_\rho - \int_0^t \Phi_0(t,\mu)B_\rho(\mu)\Psi_\rho(\mu,0)x_\rho d\mu$$

$$+ \int_0^t \Phi_0(t,\sigma)f_\rho(\sigma)d\sigma - \int_0^t\int_\sigma^t \Phi_0(t,\mu)B_\rho(\mu)\Psi_\rho(\mu,\sigma)f_\rho(\sigma)d\mu d\sigma$$

$$= \Phi_0(t,0)x_\rho + \int_0^t \Phi_0(t,\mu)f_\rho(\mu)d\mu - \{\int_0^t \Phi_0(t,\mu)B_\rho(\mu)\Psi_\rho(\mu,0)x_\rho d\mu$$

$$+ \int_0^t \Phi_0(t,\mu)B_\rho(\mu)\int_0^\mu \Psi_\rho(\mu,\sigma)f_\rho(\sigma)d\sigma d\mu\}$$

$$= \Phi_0(t,0)x_\rho - \int_0^t \Phi_0(t,\mu)[B_\rho(\mu)\Psi_\rho(\mu,0)x_\rho + B_\rho(\mu)\int_0^\mu \Psi_\rho(\mu,\sigma)f_\rho(\sigma)d\sigma$$

$$- f_\rho(\mu)]d\mu$$

$$= \Phi_0(t,0)x_\rho - \int_0^t \Phi_0(t,\mu)[B_\rho(\mu)\psi_\rho(\mu) - f_\rho(\mu)]d\mu \ ,$$

and hence $\psi_\rho(t)$ satisfies (2.8). It follows that $\psi_\rho(t)$ as given by (2.6) and (2.7) is a mild, quasi-periodic solution of equation (2.4).

Now the function G on $L^2(J,H)$ is defined by

$$G : \rho \to \psi_\rho(t) \ ,$$

where $\psi_\rho(t)$ is given by (2.6) and (2.7). If we can verify that (a) G is compact, (b) G is continuous, and (c) there exists a ball K_n in $L^2(J,H)$ such that $G(K_n) \subset K_n$, we will be able to apply Schauder's fixed point theorem to G. A fixed point $y(t)$ of G will be a required solution. Therefore we now show that (a), (b) and (c) hold.

(a) G *is compact.* Let (ρ_n) be a bounded sequence in $L^2(J,H)$. We will abbreviate $G(\rho_n)$ to ψ_n, $B(t,\zeta(\rho_n)(t))$ to $B_n(t)$,

$f(t,\rho_n(t))$ to $f_n(t)$, x_{ρ_n} to x_n and $\Psi_{\rho_n}(t,s)$ to $\Psi_n(t,s)$. We will now show that (ψ_n) contains a convergent subsequence. The functions $\psi_n(t)$ satisfy

$$\psi_n(t) = \Phi_0(t,0)x_n - \int_0^t \Phi_0(t,\sigma)[B_n(\sigma)\psi_n(\sigma) - f_n(\sigma)]d\sigma .$$

In view of I Proposition 1.4 and the remark following it, we can write this equation in the form

$$y_n = Mx_n - L(C_n y_n - f_n)$$

in $L^2(J,H)$. By I Proposition 1.3, (f_n) is a bounded sequence in $L^2(J,H)$. Using (1.12) and (2.5) in (2.7), we conclude that (x_n) is bounded in H (and hence in $L^2(J,H)$). By I Proposition 2.5 and I Theorem 2.1, the operator $L = \int_0^t \Phi_0(t,\sigma) \cdot d\sigma$ is compact in $L^2(J,H)$ and so is M (see the remark following I Proposition 1.4). However $C_n = B_n(t)$ is not a sequence of bounded linear operators in $L^2(J,H)$, but the argument and the conclusion of I Proposition 1.4 apply, if we can show that $(B_n(t)\psi_n(t))$ is a bounded sequence in $L^2(J,H)$. In order to prove this, we first note that by [17; Vol.I,p153], $\gamma < 1/2$ and Theorem 1.1, $\psi_n(t) \in \mathcal{D}(B_n(t))$ for $t > 0$ and any n. By Minkowski's inequality it follows that

$$\|B_n(t)\psi_n(t)\|_{L^2(J,H)} = \{\int_0^T \|B_n(t)\Psi_n(t,0)x_n + \int_0^t B_n(t)\Psi_n(t,\sigma)f_n(\sigma)d\sigma\|^2 dt\}^{1/2}$$

$$\le \{\int_0^T \|B_n(t)\Psi_n(t,0)x_n\|^2 dt\}^{1/2} + \{\int_0^T [\int_0^t \|B_n(t)\Psi_n(t,\sigma)f_n(\sigma)\| d\sigma]^2 dt\}^{1/2} .$$

Since $B_n(t) \in N$ it follows by (1.4), (1.6) and Theorem 1.1, that $\|B_n(t)\Psi_n(t,s)\| \le C(t-s)^{-\gamma}$, where C does not depend on n. Here we require $\gamma < 1/2$ explicitly. Therefore the first integral on the right-hand side of the above inequality is bounded by $CF(T^{1-2\gamma}/1-2\gamma)^{1/2}$. Concerning the second integral we apply Hölder's inequality

$$[\int_0^t \| B_n(t)\Psi_n(t,\sigma)f_n(\sigma)\| d\sigma]^2 \le \int_0^t \| B_n(t)\Psi_n(t,\sigma)\|^2 d\sigma \cdot \int_0^T \| f_n(\sigma)\|^2 d\sigma$$

$$\le C^2 t^{1-2\gamma}(1-2\gamma)^{-1} \cdot D^2 .$$

It follows immediately that the second integral is bounded in n, and therefore $(B_n(t)\psi_n(t))$ is a bounded sequence in $L^2(J,H)$. Hence G is compact.

(b) G *is continuous*. We use the same notation as in (a). Let $\rho_n \to \rho$ in $L^2(J,H)$. By virtue of I Proposition 1.3, $f_n(t) \to f_\rho(t)$ in $L^2(J,H)$ and by Theorem 1.7 and (1.12) we have that $\Psi_n(t,s) \to \Psi_\rho(t,s)$ strongly in H, for $s < t$ and that $\Psi_n(t,s)$ is uniformly bounded in n, s and t. It follows by (2.5) that the remaining assumptions of I Proposition 2.9 hold for $\lambda = k$. Hence by an application of I Proposition 2.9, we conclude that $(\psi_n(t))$ as given by (2.6) and (2.7) converges to $\psi_\rho(t)$ in $L^2(J,H)$, i.e. G is continuous.

(c) *Letting* $K_n = \{x \in L^2(J,H) \mid \|x\| \le n\}$, *there exists an* n *such that* $G(K_n) \subset K_n$. Supposing this is not so, then there exists a sequence (x_ℓ) in $L^2(J,H)$ such that $x_\ell \in K_\ell$ and $G(x_\ell) \notin K_\ell$, i.e. $1 < \ell^{-2}\|G(x_\ell)\|^2$ for all $\ell \in \mathbb{N}$. Combining (2.6) and (2.7) and using (1.12) and (2.5) we obtain a contradiction with (2.3) just as in the proof of I Theorem 3.1 (Part (c)).

Since G satisfies (a), (b), (c) we know that it has a fixed point $y(t)$. By virtue of (2.6) we have

$$y(t) = \Psi_y(t,0)x_y + \int_0^t \Psi_y(t,\sigma)f_y(\sigma)d\sigma .$$

Since $\Psi_y(t,s)$ is strongly continuous and uniformly bounded, one verifies, just as in the proof of I Theorem 3.1, that $y(t)$ is continuous in J.

As it was shown in (2.8), $y(t)$ satisfies (2.2) and of course $ky(0) = y(T)$. Therefore a fixed point $y(t)$ of G is a mild, quasi-periodic solution of (2.1). This completes the proof of Theorem 2.1 $\square$.

Remark. We note that the conditions on $f(t,x)$ are the same as in Chapter I Theorem 3.1 and refer to the Remark following this theorem. An example of such a function is given at the end of Section 4 of Chapter I.

We saw that a fixed point of G is a continuous function. If $f(t,y(t))$ is Hölder continuous, then $\psi_y(t)$ as given in (2.6) is differentiable. Therefore by Theorems 1.1 and 2.1 we obtain

Corollary 2.2. *In addition to the hypothesis of* Theorem 2.1, *suppose that for every* $y(t) \in C(J,H)$, $f(t,y(t))$ *is Höldercontinuous. Then equation* (2.1) *has a strict, quasi-periodic solution.*

Remark. This additional assumption holds if we suppose, for example, that $f(t,x)$ is Höldercontinuous in both t and x and if we consider the function $f(t,\zeta(y)(t))$ in (2.1) instead of $f(t,y(t))$.

In the next section an example will be given illustrating the result of Theorem 2.1.

3. EXAMPLE

In this section we will illustrate the results obtained in Section 2. It will be shown that the nonlinear partial differential equation (3.5) with variable boundary conditions (3.6) on a rectangle, has a quasi-periodic solution.

We will split this section into two parts. In the first one a time dependent Sturm-Liouville operator $A(t)$ with variable domain will be introduced as will an unbounded perturbation $B(t,z)$, represented by a multiplication operator $p(t,z) \in L^2(a,b)$. We will state the main result (Theorem 3.1) and verify condition (AII.2) $(\gamma<1/2)$ and the continuity of $B(t,z)(A(t)-\lambda I)^{-1}$. Condition (AII.2) $(\gamma<1/2)$, with the perturbation described above did not permit us to consider partial differential operators. We refer to the Remark following Theorem 3.1.

The second part of the section will be concerned with the Höldercontinuity of $B(t,\zeta(\rho))A(t)^{-1}$. It leads to some interesting problems relating to the dependence of the eigenvalues and eigenfunctions on the time dependent Sturm-Liouville system (3.11), (3.12), (3.13).

3.1 An example of an operator $A(t)$ and a perturbation $B(t,\zeta(\rho))$.

For the following we set $H = L^2(a,b)$, ((a,b) a finite interval). For each $t \in J$ denote with $-A(t)$ the operator in H defined by the differential form

$$\ell_t(y) = y'' - q(t)y \;, \quad ('=d/dx) \tag{3.1}$$

with domain

$$\mathcal{D}(-A(t)) = \{y \in H | y,\ y' \text{ are absolutely continuous, } \ell_t(y) \in H \text{ and } \gamma(t)y(a) - y'(a) = 0,\ \delta(t)y(b) - y'(b) = 0\} \tag{3.2}$$

Here we assume that the coefficients $q(t)$, $\gamma(t)$, $\delta(t)$ are real-valued and that their derivatives are Hōldercontinuous in J .

$-A(t)$ represents the classical Sturm-Liouville operator. It is a self-adjoint, linear operator in H whose resolvent is compact, see e.g. Neumark [61; pp172,176,189,195,200].

The eigenvalues $\lambda_n(t)$ of $A(t)$ are bounded below as [61; p203] asserts, and they tend to infinity with n , for every $t \in J$. It is possible to choose

$$0 < q_1 \le q(t) \le q_2 \tag{3.3}$$

where q_1 is sufficiently large so that the eigenvalues $\lambda_n(t)$ exceed a fixed positive number. This can be proved with arguments similar to [12; p213]. In the sequel we assume that q_1 is sufficiently large.

It is well known that for each $t \in J$ the eigenfunctions $y_n(x,t)$ corresponding to $\lambda_n(t)$ form an orthonormal basis of H , see e.g. Hille [31; p398,409]. Thus for $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \le 0$, and $f \in H$

$$((A(t)-\lambda I)^{-1}f)(x) = \sum_n (f_n(t)/(\lambda_n(t)-\lambda))y_n(x,t) \ , \tag{3.4}$$

where $f_n(t)$ denotes the n.th Fourier coefficient of f for t . It follows that

$$\|(A(t)-\lambda I)^{-1}\| \le M/(1+|\lambda|) \ ,$$

for some constant M . Referring to Tanabe [72; p140] one can now conclude that the operator $A(t)$ satisfies the assumptions stated in

Section 5.3 of the same book [72; p129,130]. Therefore $A(t)$ satisfies conditions (AI.1) - (AI.4) and (AII.1).

Concerning the perturbation $B(t,z)$, let p be a map from $J \times H$ into H. We assume that

(i) $p(t,z)$ is continuous in z for every $t \in J$.

For $\rho(t) \in L^2(J,H) = L^2(J\times(a,b))$, see Balakrishnan [3; p134], we now consider $p(t,\zeta(\rho)(t))$, where ζ is defined as in (1.10). In the following we will abbreviate $p(t,\zeta(\rho)(t))$ to $p_\rho(t)$ or $p_\rho(x,t)$. Further we suppose that

(ii) $\|p_\rho(t)\|_H \le K'$, where K' is independent of ρ and t.

(iii) For any $\rho \in L^2(J,H)$, $p_\rho(t)$ is Höldercontinuous in t, i.e.

$$\|p(t,\zeta(\rho)(t)) - p(s,\zeta(\rho)(s)\|_H \le K_\rho |t-s|^\beta .$$

The linear operator $B(t,\zeta(\rho))$ in H is defined by multiplication

$$f(x) \to p_\rho(x,t) \cdot f(x)$$

It is in general unbounded, since for given ρ and t, the image of $p_\rho(x,t)$ under this operator does not belong to $L^2(a,b)$ unless $p_\rho(x,t) \in L^4(a,b)$. Moreover it is closed and its domain contains that of $A(t)$ for every $\rho \in L^2(J,H)$ and $t \in J$. The two statements are now proved:

$\mathcal{D}(A(t)) \subset \mathcal{D}(B(t,\zeta(\rho)))$ for any $\rho \in L^2(J,H)$ and $t \in J$, since the continuous functions on J are contained in the domain of $B(t,\zeta(\rho))$.

The operator $B(t,\zeta(\rho))$ *is closed.* For given ρ and t, we abbreviate $p_\rho(t)$ to $p(x)$. The function $p(x)$ belongs to $L^2(a,b)$. Let now $f_n \to f$ and $pf_n \to g$ in $L^2(a,b)$. We can therefore find a subsequence (f_ℓ) such that $|f_\ell(x) - f(x)|$ and hence also $|p(x)f_\ell(x) - p(x)f(x)|$

converge to zero for almost every $x \in (a,b)$. By Fatou's lemma one obtains

$$\|pf\| = \left(\int_a^b \liminf_{\ell} |p(x)f_\ell(x)|^2 dx\right)^{1/2} \le \left(\liminf_{\ell} \int_a^b |p(x)f_\ell(x)|^2 dx\right)^{1/2}$$

$$= \liminf_{\ell} \|pf_\ell\| \le \lim_{\ell} \|pf_\ell - g\| + \|g\| < \infty .$$

Hence f belongs to the domain of $B(t,\zeta(\rho))$. It remains to show that $g = pf$ in $L^2(a,b)$. From the assumptions it follows that $pf_n \to pf$ in $L^1(a,b)$ and $pf_n \to g$ in $L^1(a,b)$. Since $pf \in L^2(a,b)$ the results follows □

We are now in a position to state the main result.

Theorem 3.1. *Let the operator* $A(t)$ *be given by* (3.1), (3.2) *and assume that the function* $p(t,z)$ *satisfies* (i), (ii), (iii) *above. Then, if* $f(t,z)$ *is as in* Corollary 2.2, *the non-linear partial differential equation*

$$\partial u/\partial t - \partial^2 u/\partial x^2 + (q(t)+p(x,t,\zeta(u)))u = f(x,t,u) \qquad (3.5)$$

with boundary conditions

$$\begin{aligned} \gamma(t)u(a,t) - \partial u(a,t)/\partial x &= 0 \qquad t \in (0,T] \\ \delta(t)u(b,t) - \partial u(b,t)/\partial x &= 0 \qquad t \in (0,T] \qquad (3.6) \\ ku(x,0) = u(x,T) &\quad \text{a.e. in } x \in (a,b) \end{aligned}$$

has a strict solution, where k *is given by* Theorem 2.1.

We will first prove that the operators $A(t)$ and $B(t,\zeta(\rho))$ satisfy (AII.2) as well as the continuity assumption in Theorem 2.1. The proof of the Höldercontinuity of $B(t,\zeta(\rho))A(t)^{-1}$ (see Theorem 3.6) will be postponed until after we have investigated the Sturm-Liouville system (3.11), (3.12), (3.13).

Remark. Suppose that we replace the operator $A(t)$ as given in (3.1), (3.2) by the Laplace operator on the unit circle together with the third boundary condition $\partial u/\partial r + h(t)u = 0$, c.f.[59; p103], but retain the same perturbation operator (defined by multiplication). We will show that under these circumstances condition (AII.2), $\gamma < 1/2$, is not satisfied.

In fact, by the method used in the proof of (AII.2) below, (AII.2) leads us to the double series

$$\Sigma(x_{n\ell})^{-4\gamma}, \tag{3.7}$$

where $x_{n\ell}$ is the ℓ-th root of the equation $xJ_n'(x) + hJ_n(x) = 0$, and J_n denotes the n-th Bessel function. The series in (3.7) diverges already for $\gamma = 1/2$. In order to see this, let $y_{n\ell}$ denote the ℓ-th root of $J_n(x) = 0$. By using the product representation, see Watson [81; p498], it follows that the double series $\Sigma(y_{n\ell})^{-2}$ already diverges*. Since by Dixon's theorem [81; p480] the positive roots $x_{n\ell}$ are interlaced with $y_{n\ell}$, the result follows. Therefore the restriction $\gamma < 1/2$ in (AII.2) limits $A(t)$ to ordinary differential operators, in the case of the given perturbation.

Proof of (AII.2). We already know that $B(t,\zeta(\rho))$ is a closed linear operator whose domain contains that of $A(t)$, for every ρ and t. It will now be verified that for given positive constants M_1 and $\gamma < 1/2$

$$\|\lambda^{1-\gamma}B(t,\zeta(\rho))(A(t)-\lambda I)^{-1}\| \leq M_1$$

for any $\rho \in L^2(J,H)$, $t \in J$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda \leq 0$. Let $f \in H$.

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*I am grateful to Professor W.Magnus who has pointed this out to me.

With (3.4) we are lead to consider

$$W = \| |\lambda|^{1-\gamma} p_\rho(t) \sum_n (f_n(t)/(\lambda_n(t)-\lambda)) y_n(x,t) \|_H^2$$

$$\leq \int_a^b \{\sum_n |f_n(t)| |p_\rho(x,t) y_n(x,t)| \lambda|^{1-\gamma}/(\lambda_n(t)-\lambda)|\}^2 dx$$

$$\leq \|f\|_H^2 \int_a^b |p_\rho(x,t)|^2 \sum_n |y_n(x,t)|\lambda|^{1-\gamma}/(\lambda_n(t)-\lambda)|^2 dx \ ,$$

with Hölder's inequality. Since $q(t)$ is bounded it follows that $y_n(x,t)$ is bounded uniformly in n, x, t, c.f. Tricomi [75; p168], i.e.

$$|y_n(x,t)| \leq S \ . \tag{3.8}$$

By assumption (ii) on $p_\rho(t)$, we therefore obtain

$$W \leq K'^2 S^2 \|f\|_H^2 \cdot \sum_n (|\lambda|^{1-\gamma}/|\lambda_n(t)-\lambda|)^2 \ .$$

It thus remains to show that the above series is finite for any λ in the half-plane $\text{Re}\,\lambda \leq 0$.

We fix n and t for the moment and set $a = \lambda_n(t)$ which is positive by the assumption on q_1. The function

$$Q(\lambda) = |\lambda|^{1-\gamma}/|\lambda-a| = |\lambda|^{-\gamma}|1-a/\lambda|^{-1}$$

will be majorized by a function P that attains a maximum given by $Ra^{-\gamma}$.

Let $\lambda = re^{i\psi}$, $\pi/2 \leq \psi \leq 3\pi/2$. $Q(\lambda)$ can be written as

$$Q(\lambda) = r^{1-\gamma}(r^2-2ar\cos\psi+a^2)^{-1/2} \ .$$

Since $\cos\psi \leq 0$ in the domain of ψ, it follows that

$$Q(\lambda) \leq r^{1-\gamma}(r^2+a^2)^{-1/2} = P(r) \ ,$$

for all λ, Re $\lambda \le 0$. Clearly $P(r) \to 0$ if $r \to \infty$ or if $r \to 0$. With

$$dP/dr = -\gamma r^{-\gamma}(r^2+a^2)^{-3/2}(r+a((1-\gamma)/\gamma)^{1/2})(r-a((1-\gamma)/\gamma)^{1/2})$$

it is easily seen that $P(r)$ attains its only maximum at $r_0 = a((1-\gamma)/\gamma)^{1/2}$, and

$$P(r_0) = a^{-\gamma}R(\gamma) \ , \quad \text{where}$$

$$R(\gamma) = (1-\gamma)^{1/2}((1-\gamma)/\gamma)^{-\gamma/2} \ .$$

Therefore for all λ with Re $\lambda \le 0$,

$$Q(\lambda) \le a^{-\gamma}R(\gamma) \ .$$

We can thus majorize our series, i.e.

$$\sum_n (|\lambda|^{1-\gamma}/|\lambda_n(t)-\lambda|)^2 \le R(\gamma)^2 \sum_n (\lambda_n(t))^{-2\gamma} \ . \tag{3.9}$$

Thus an estimate on the eigenvalues $\lambda_n(t)$ of $A(t)$ is required.

Along with

$$y'' + (\lambda(t)-q(t))y = 0$$

we consider the two equations, see (3.3),

$$y'' + (\lambda_1(t)-q_1)y = 0 \ , \ y'' + (\lambda_2(t)-q_2)y = 0 \ ,$$

together with the same boundary conditions that appear in (3.2). For a fixed $t \in J$, we have from the comparison theorem (see for example Tricomi [75; p125-129]),

$$\lambda_{1,n}(t) \le \lambda_n(t) \le \lambda_{2,n}(t) \qquad n \ge 0 \ ,$$

$\lambda_{1,n}(t)$, $\lambda_{2,n}(t)$ denoting the (n+1)th eigenvalues of these equations. By methods similar to [75; p118], it follows that the eigenvalues $\lambda_n(t)$ tend to infinity uniformly in J. We can therefore find an integer n_0 such that for $n \geq n_0$, $q_2 < \lambda_n(t)$, and hence conclude with [75; p128] that

$$0 < \lambda_n(\min) = ((n-1)\pi/(b-a))^2 + q_1 < \lambda_n(t) \leq ((n+1)\pi/(b-a))^2 + q_2 , \qquad (3.10)$$

for $n \geq n_0$. From this it follows that the series in (3.9) converge, since we can choose $\gamma \in (1/4,1/2)$. Hence (AII.2) is verified □ .

<u>Proof of the continuity of $B(t,z)(A(t)-\lambda I)^{-1}$</u> . For given $t \in J$, $\lambda \in \mathbb{C}$, with $\mathrm{Re}\,\lambda \leq 0$ and $f \in H$, we let $z_j \to z$ in H . Because of (3.4) we obtain

$$\int_a^b |[p(x,t,z_j) - p(x,t,z)]\sum_n (f_n(t)/(\lambda_n(t)-\lambda))y_n(x,t)|^2 dx$$

$$\leq \int_a^b |p(x,t,z_j) - p(x,t,z)|^2 \{\sum_n |f_n(t)| \cdot S/|\lambda_n(t)-\lambda|\}^2 dx$$

$$\leq \|f\|^2 S^2 \sum_n |\lambda_n(t) - \lambda|^{-2} \int_a^b |p(x,t,z_j) - p(x,t,z)|^2 dx$$

by Hölder's inequality and (3.8). In view of (3.3), (3.10) and the assumption (i) on $p(t,z)$ the result follows □ .

In order to complete the proof of Theorem 3.1, we only need to verify that $B(t,\zeta(\rho))A(t)^{-1}$ is Höldercontinuous. This we do in Part 3.2.

3.2 <u>The Höldercontinuity of $B(t,\zeta(\rho))A(t)^{-1}$</u> .

For this purpose we investigate the dependence on the parameter t , of the eigenvalues and eigenfunctions of the Sturm-Liouville system

$$y'' + (\lambda - q(x,t))y = 0 , \quad ' = d/dx , \ a \leq x \leq b \qquad (3.11)$$

$$y(a)\cos\alpha(t) - y'(a)\sin\alpha(t) = 0 , \quad 0 \leq \alpha(t) < \pi \qquad (3.12)$$

$$y(b)\cos\beta(t) - y'(b)\sin\beta(t) = 0 , \quad 0 < \beta(t) \leq \pi \qquad (3.13)$$

where α , β are real-valued, continuously differentiable functions on $J = [0,T]$, and $q(x,t) \in C^1([a,b]\times[0,T])$. We note that the boundary conditions that appear in the definition of $A(t)$, (3.2), are a special case of (3.12), (3.13) (see [75; p109]).

In Theorem 3.2 we will show that the eigenvalues λ_n are continuously differentiable functions of t . If we drop the dependence of $q(x,t)$ on x , then $d\lambda_n/dt$ can be given explicitly. Moreover $(\lambda_n(t))^{-1}d\lambda_n/dt$ is uniformly bounded in $n \in \mathbb{N}$ and $t \in J$. This is the content of Theorem 3.3. With these results available, we will be able to prove the required Höldercontinuity in Theorem 3.6.

It is well known of course how the eigenvalues λ_n change as the coefficient function $q(x)$ varies and the boundary conditions are kept constant, c.f. Titchmarsh [73] for the singular case. One can conclude with Courant, Hilbert [15; p419] that the n-th eigenvalue λ_n depends continuously on the functions α , β in (3.12), (3.13) and q in (3.11), but the change depends on n .

Systems with the spectral parameter also in the boundary conditions have been considered since the beginning of this century, see Fulton [27] and the references contained therein. No results with variable boundary conditions of type (3.12), (3.13) seem to have been obtained so far, c.f. [68].

Theorem 3.2. *The eigenvalues* λ_n *of the Sturm-Liouville system* (3.11), (3.12), (3.13) *are continuously differentiable functions of* t *in* $[0,T]$. *The derivative* $d\lambda_n/dt$ *is given by* (3.19).

Proof. To establish this result problem (3.11), (3.12) will be transformed with Prüfer's method in a first step. A well known result concerning the dependence on parameters and initial conditions will then be applied to yield expressions for the partial derivatives of the solution. Lastly the implicit function theorem will take care of the second boundary condition (3.13) and will give us $d\lambda_n/dt$. Accordingly Prüfer's transformation

$$y(x,\lambda,t) = \rho(x,\lambda,t)\sin\psi(x,\lambda,t) ,$$

$$\partial y/\partial x \quad = \rho(x,\lambda,t)\cos\psi(x,\lambda,t) ,$$

applied to (3.11), (3.12) leads us to the equation

$$\psi' = \cos^2\psi(x,\lambda,t) + (\lambda - q(x,t))\sin^2\psi(x,\lambda,t) , \quad a \le x \le b \tag{3.14}$$

$$\psi(a,\lambda,t) = \alpha(t) \tag{3.15}$$

It is well known that for any given $t \in [0,T]$ and $\lambda \in \mathbb{R}$ equation (3.14) has a unique solution in $[a,b]$ that satisfies (3.15). And if $\psi(x,\lambda,t)$ satisfies (3.15), then $y(x,\lambda,t)$ satisfies (3.12). For given t, $y(x,\lambda,t)$ will satisfy the second boundary condition (3.13) if and only if λ_n can be found so that

$$\psi(b,\lambda_n,t) = \beta(t) + n\pi , \quad n \in \mathbb{N} \cup \{0\} . \tag{3.16}$$

Now for fixed t, $\psi(b,\lambda,t)$ is an increasing function of λ that takes on every positive value exactly once. Therefore there exists a unique λ_n for every n. The λ_n are of course our eigenvalues. They form a sequence $\lambda_0<\lambda_1<\lambda_2< \ldots$ which tends to infinity, see e.g. Hille [31; pp395-399].

We will use equations (3.14), (3.15), (3.16) to investigate λ_n as a function of t. With the transformation

$$\psi(x,\lambda,t) = \theta(x,\lambda,t) + \alpha(t)$$

equations (3.14) and (3.15) become

$$\theta' = \cos^2(\theta(x,\lambda,t)+\alpha(t)) + (\lambda-q(x,t))\sin^2(\theta(x,\lambda,t)+\alpha(t)) , \ a \le x \le b \qquad (3.17)$$

$$\theta(a,\lambda,t) = 0 .$$

Equation (3.17) contains two parameters and can be written as $\theta' = f(x,\theta,\lambda,t)$.

Let $\phi(x,\lambda_0,t_0)$ be a solution of (3.17) satisfying $\phi(a,\lambda_0,t_0) = 0$ for some $\lambda_0 \in \mathbb{R}$ and $t_0 \in [0,T]$. Then from Coppel [14; p.22] we know that for any $x \in [a,b], \partial\phi(x,\lambda_0,t_0)/\partial\lambda$ exists, is continuous in all its arguments and is the solution of

$$y' = f_\theta(x,\phi(x,\lambda_0,t_0),\lambda_0,t_0)y + f_\lambda(x,\phi(x,\lambda_0,t_0),\lambda_0,t_0), \quad a \le x \le b$$

$y(a) = 0$, where $f(x,\theta,\lambda,t)$ denotes the right hand side of (3.17).

Similarly let $\phi(x,\lambda_0,t_0)$ be a solution of (3.17) for which $\phi(a,\lambda_0,t_0) = 0$ for some $\lambda_0 \in \mathbb{R}$ and $t_0 \in [0,T]$. Then for any $x \in [a,b]$ $\partial\phi(x,\lambda_0,t_0)/\partial t$ exists, is continuous in all its arguments and is the solution of

$$y' = f_\theta(x,\phi(x,\lambda_0,t_0),\lambda_0,t_0)y + f_t(x,\phi(x,\lambda_0,t_0),\lambda_0,t_0), \quad a \le x \le b$$

$y(a) = 0$.

According to the transformation used $\phi(x,\lambda,t) = \psi(x,\lambda,t) - \alpha(t)$ and thus $\partial\psi/\partial\lambda = \partial\phi/\partial\lambda$, $\partial\psi/\partial t = \partial\phi/\partial t + d\alpha/dt$. It now follows that

$$\frac{\partial\phi}{\partial\lambda}(b,\lambda_0,t_0) = \int_a^b \exp\{\int_s^b \sin(2\psi(x,\lambda_0,t_0))(\lambda_0-q(x,t_0)-1)dx\} \cdot \sin^2\psi(s,\lambda_0,t_0)ds \qquad (3.18)$$

This expression is positive for any λ_0 , t_0 , since the integrand is non-negative and the zeros of ψ are isolated.

Now the second boundary condition (3.13) in the form of equation (3.16) is taken into account by introducing for each $n \in \mathbb{N} \cup \{0\}$ the function $G_n(\lambda,t)$ defined for any $t \in [0,T]$ and $\lambda \in \mathbb{R}$ by

$$G_n(\lambda,t) = \psi(b,\lambda,t) - \beta(t) - n\pi ,$$

where $\psi(x,\lambda,t)$ satisfies (3.14) and (3.15). Then for a given $t_0 \in [0,T]$ $G_n(\lambda,t_0) = 0$ if $\lambda = \lambda_n(t_0)$ the n-th eigenvalue corresponding to t_0 . Moreover

$$\partial G_n/\partial\lambda = \partial\psi(b,\lambda,t)/\partial\lambda \quad \text{and} \quad \partial G_n/\partial t = \partial\psi(b,\lambda,t)/\partial t - d\beta/dt$$

are continuous in t and λ . From (3.18) it follows that for any $n \in \mathbb{N} \cup \{0\}, \partial G_n(\lambda_n(t_0),t_0)/\partial\lambda \neq 0$.

The implicit function theorem and the above results now guarantee that for each $n \in \mathbb{N} \cup \{0\}$, $\lambda_n(t) \in C^1[0,T]$ and

$$\frac{d\lambda_n}{dt} = -\frac{\partial G_n/\partial t}{\partial G_n/\partial\lambda} \tag{3.19}$$

$$= \frac{d\beta/dt - d\alpha/dt - \int_a^b \exp(F(s)) \cdot (d\alpha/dt) \cdot \sin(2\psi(s,\lambda_n(t),t))(\lambda_n(t)-q(s,t)-1)ds}{\int_a^b \exp(F(s)) \cdot \sin^2\psi(s,\lambda_n(t),t)ds}$$

$$+ \frac{\partial q}{\partial t}(\xi,t) \quad \text{for some} \quad \xi = \xi(t) \in (a,b) ,$$

where

$$F(s) = F(s,n,t) = \int_s^b \sin(2\psi(x,\lambda_n(t),t))(\lambda_n(t)-q(x,t)-1)dx \tag{3.20}$$

This completes the proof of Theorem 3.2 □ .

We consider now the behaviour of $d\lambda_n/dt$ for large n . It is possible to evaluate explicitly, the expression given in (3.19) for $d\lambda_n/dt$ and to estimate its behaviour as $n \to \infty$, provided $q(x,t)$ is independent of x . *Thus we assume from here on again that* $q(t) \in C^1[0,T]$.

Theorem 3.3. *If* $q(t) \in C^1[0,T]$ *depends only upon* t *, then for* n *sufficiently large (see* (3.22)) $d\lambda_n/dt$ *is explicitly given by* (3.28) *and*

$$(\lambda_n(t))^{-1}(d\lambda_n/dt)$$

are uniformly bounded in $n \geq n_0$ *and* $t \in [0,T]$.

Proof. The proof is tedious and involves mainly the evaluation of the integrals occurring in (3.19). Accordingly setting

$$\lambda - q(t) = \tau^2(\lambda,t) \quad \text{and} \quad \lambda_n(t) - q(t) = \tau_n(t)^2$$

for λ sufficiently large, it follows from (3.14), c.f. Tricomi [75; p115] that

$$\tan\psi(x,\lambda,t) = \tau^{-1}\tan\tau\cdot(x-c(t)) \tag{3.21}$$

where in the light of (3.15) we take $c(t)$ so that

$$\tau\cdot(a-c(t)) = \arccos\{\cos\alpha(t)/(\cos^2\alpha(t)+\tau^2\sin^2\alpha(t))^{1/2}\} \in [0,\pi) .$$

Since $\lambda_n(t)$ tends uniformly to infinity in $[0,T]$ as $n \to \infty$, then so does $\tau_n(t)$. Therefore we choose n large enough in the sequel so that the following inequality holds for any $t \in [0,T]$.

$$\pi/2(b-a) \leq (\lambda_n(t) - q(t))^{1/2} \tag{3.22}$$

Now let n be given and sufficiently large and let $t = t_0$ be fixed. Thus $\tau_n(t_0) = \tau_n$ and the constant in (3.21) $c(t_0) = c_0$ are determined. As is seen from (3.14), $\psi(x,\lambda,t)$ is an increasing function of x . Its behaviour is described in [75; p115].

We subdivide the interval $[a,b]$ according to the following rule. Let n be given by (3.22). Then there exist integers ℓ and m , where $\ell \in \{0,1,2\}$ and depending on t and $m \geq 0$ (see (3.21)) such that

$$c_0 + \pi(\ell-1)/2\tau_n < a \le c_0 + \pi\ell/2\tau_n \le \,..$$

$$..\le c_0 + \pi(\ell+2m)/2\tau_n \le b < c_0 + \pi(\ell+2m+2)/2\tau_n \qquad (3.23)$$

$F(s)$ of (3.20) now becomes

$$F(s) = (\tau_n{}^2-1)\int_s^b \sin\{2\arctan[\tau_n^{-1}\tan\tau_n(x-c_0)]\}dx$$

Notice that the integrand vanishes for $x = c_0 + \pi m/2\tau_n$, m an integer. Further

$$d\lambda_n/dt = dq/dt + \qquad (3.24)$$

$$+\frac{d\beta/dt-d\alpha/dt-(d\alpha/dt)(\tau_n{}^2-1)\int_a^b \exp(F(s))\sin\{2\arctan[\tau_n^{-1}\tan\tau_n(s-c_0)]\}ds}{\int_a^b \exp(F(s))\sin^2\{\arctan[\tau_n^{-1}\tan\tau_n(s-c_0)]\}ds}\ .$$

In evaluating $F(s)$ we distinguish between the two cases:

$$\alpha)\quad a \le s \le c_0 + \pi(\ell+2m)/2\tau_n$$

or

$$\beta)\quad c_0 + \pi(\ell+2m)/2\tau_n < s \le b$$

In case (α) we can find an integer k such that $0 \le c_0 + \pi(\ell+2k)/2\tau_n +$ $- s < \pi/\tau_n$ and split the integral $F(s)$ into three parts, namely, $F(s) \equiv I + II + III$,

I representing the integral with the limits s and $c_0 + \pi(\ell+2k)/2\tau_n$

II representing the integral with the limits $c_0 + \pi(\ell+2k)/2\tau_n$ and $c_0 + \pi(\ell+2m)/2\tau_n$

III representing the integral with the limits $c_0 + \pi(\ell+2m)/2\tau_n$ and b .

With the help of the substitution

$$v = (\tau_n)^{-1}\tan\tau_n(x-c_0) \qquad (3.25)$$

the indefinite integral $J(v)$ of $F(s)$ becomes

$$J(v) = \frac{(\tau_n)^2-1}{(\tau_n)^2}\int \frac{2v}{(v^2+1)(v^2+(\tau_n)^{-2})}\,dv = \log\frac{v^2+(\tau_n)^{-2}}{v^2+1} \quad .$$

By considering subintervals, integral II is readily seen to equal zero. As to the integrals I and III we have to distinguish between ℓ (in α or β) odd or even. The case ℓ even in integral I is considered in detail now. All the other arguments are very similar. Here then two situations can occur.

i) $c_0 + \pi(\ell+2k-1)/2\tau_n \le s \le c_0 + \pi(\ell+2k)/2\tau_n$

ii) $c_0 + \pi(\ell+2k-2)/2\tau_n < s < c_0 + \pi(\ell+2k-1)/2\tau_n$

As x runs through the interval $[s, c_0+\pi(\ell+2k)/2\tau_n]$ then in (i) v runs through $[(\tau_n)^{-1}\tan\tau_n(s-c_0), 0]$. But in (ii) the corresponding v-interval consists of two parts, namely v runs through $[(\tau_n)^{-1}\tan\tau_n(s-c_0), \infty)$ and through $(-\infty,0]$. In (ii) one thus obtains for the integral I

$$\lim(z\to\infty)J(z) - \lim(z\to-\infty)J(z) + J(0) - J((\tau_n)^{-1}\tan\tau_n(s-c_0)) \ .$$

But clearly $\lim(z\to\pm\infty)J(z) = 0$. Thus in both (i) and (ii) when ℓ is even we obtain for the integral I

$$\log\frac{\tan^2\tau_n(s-c_0) + (\tau_n)^2}{(\tau_n)^2(\tan^2\tau_n(s-c_0)+1)}$$

if $\tau_n(s-c_0)$ is not an odd multiple of $\pi/2$, and $\log(\tau_n)^{-2}$ otherwise. Arguing in this way we obtain one expression for $\exp(F(s))$, despite the different cases that occur.

$$\exp(F(s)) = Tb\,\frac{\tan^2\tau_n(s-c_0) + (\tau_n)^2}{\tan^2\tau_n(s-c_0)+1}$$

if $\tau_n(s-c_0)$ is not an odd multiple of $\pi/2$, and Tb otherwise, where

$$\mathrm{Tb} = \frac{\tan^2\tau_n(b-c_0) + 1}{\tan^2\tau_n(b-c_0) + (\tau_n)^2}$$

if $\tau_n(b-c_0)$ is not an odd multiple of $\pi/2$, and 1 otherwise.

Next we describe how to evaluate the numerator N of (3.24),

$$N = ((\tau_n)^2-1)\int_a^b \exp(F(s))\sin\{2\arctan[(\tau_n)^{-1}\tan\tau_n(s-c_0)]\}ds .$$

Its integrand is again zero for $s = c_0 + \pi m/2\tau_n$. Thus the subdivision of (3.23) again leads to the three parts of the integral N, namely

$N \equiv I + II + III$

I representing the integral with the limits a and $c_0 + \pi\ell/2\tau_n$

II representing the integral with the limits $c_0 + \pi\ell/2\tau_n$ and $c_0 + \pi(\ell+2m)/2\tau_n$

III representing the integral with the limits $c_0 + \pi(\ell+2m)/2\tau_n$ and b.

With the substitution (3.25) the indefinite integral $J(v)$ of N becomes

$$J(v) = \mathrm{Tb}\frac{(\tau_n)^2-1}{(\tau_n)^2}\int\frac{2v}{(v^2+(\tau_n)^{-2})^2}dv = -\mathrm{Tb}\frac{(\tau_n)^2-1}{(\tau_n)^2}\cdot\frac{1}{v^2+(\tau_n)^{-2}} .$$

With similar arguments as before, we see that integral II is zero and thus obtain for N

$$N = \mathrm{Tb}((\tau_n)^2-1)\sin\tau_n(b-a)\cdot\sin\tau_n(a+b-2c_0) \tag{3.26}$$

Lastly the integral D in the denominator of (3.24) is considered. Its integrand is zero for $s = c_0 + \pi m/\tau_n$. Therefore the integral D may be split into three parts, but slightly differently to (3.23). More explicitly, we can find a non-negative integer k satisfying $0 \le b - c_0 + \pi(\ell+k)/2\tau_n < \pi/2\tau_n$ such that $D \equiv I + II + III$ where

I represents the integral with limits a and $c_0 + \pi\ell/2\tau_n$

II represents the integral with limits $c_0 + \pi\ell/2\tau_n$ and $c_0 + \pi(\ell+k)/2\tau_n$

III represents the integral with limits $c_0 + \pi(\ell+k)/2\tau_n$ and b.

With the substitution $\phi = \tau_n(s-c_0)$ the indefinite integral $J(v)$ of D becomes

$$J(v) = (Tb/\tau_n)\int \sin^2\phi \, d\phi = (Tb/4\tau_n)(2\phi - \sin 2\phi) .$$

Thus with similar arguments we obtain

$$D = (Tb/2\tau_n)(\tau_n(b-a) - \sin\tau_n(b-a)\cos\tau_n(a+b-2c_0)) . \tag{3.27}$$

From (3.24), (3.26), (3.27), we now have

$$d\lambda_n/dt = (d\beta/dt - d\alpha/dt[1+N])/D + dq/dt \quad \text{at} \quad t = t_0. \tag{3.28}$$

For m smaller than the value n determined by (3.22) the $d\lambda_m/dt$ are of course uniformly bounded since $\lambda_n(t) \in C^1[0,T]$. $d\beta/dt$, $d\alpha/dt$, dq/dt are uniformly bounded in $t \in [0,T]$. Hence to complete the proof it remains to show that

$$(\lambda_n(t_0)D)^{-1} \quad \text{and} \quad N/\lambda_n(t_0)D$$

are uniformly bounded in n and t. By (3.22)

$$D \geq (Tb/2)((b-a) - (\tau_n)^{-1}) \geq Tb(b-a)(\pi-1)/2\pi$$

and thus replacing $(\tau_n)^2$ by $\lambda_n(t_0) - q(t_0)$ we obtain

$$\left|\lambda_n(t_0)D\right|^{-1} \leq \frac{2\pi}{(\pi-1)(b-a)\lambda_n(t_0)}\Big|1 +$$

$$+ (\lambda_n(t_0)-q(t_0)-1)\cos^2\{(\lambda_n(t_0)-q(t_0))^{1/2}(b-c(t_0))\}\Big|$$

and

$$\left|N/\lambda_n(t_0)D\right| \leq \frac{2\pi}{(\pi-1)(b-a)\lambda_n(t_0)}\left|\lambda_n(t_0) - q(t_0) - 1\right|$$

which both have the desired boundedness properties. This completes the proof □ .

We need the next two auxiliary results to establish the Höldercontinuity. In them $\lambda_n(t)$ and $y_n(x,t)$ again denote the eigenvalues and eigenfunctions respectively of the Sturm-Liouville system (3.11), (3.12), (3.13).

Theorem 3.4. *If* $q(t)$ *belongs to* $C^1[0,T]$ *and satisfies* (3.3), *then*

$$(\lambda_n(t))^{-1/2}\partial y_n(x,t)/\partial t$$

is uniformly bounded in x, t *and* $n \geq 0$.

Proof. We can find $n_0 \in \mathbb{N}$ such that for any $t \in J$ and $n \geq n_0$ $q(t) < \lambda_n(t)$. Since for each $n \geq 0$, $\partial y_n(x,t)/\partial t$ is continuous in $x \in [a,b]$ and $t \in J$, and because of (3.3), $(\lambda_n(t))^{-1/2}(\partial y_n/\partial t)$ is uniformly bounded in x, t and $n < n_0$.

For $n \geq n_0$ we set $\mu_n = \mu_n(t) = (\lambda_n(t)-q(t))^{1/2}$. Since q is independent of x the non-normalized eigenfunctions of (3.11), (3.12), (3.13) have the form

$$y_n(x,t) = \sin\alpha(t)\cdot\cos\mu_n(x-a) + (\cos\alpha(t)/\mu_n)\cdot\sin\mu_n(x-a)$$

Their derivatives w.r.t. t are given as

$$\begin{aligned}\partial y_n/\partial t = {} & \cos\mu_n(x-a)\{(d\alpha/dt)\cos\alpha(t) + (x-a)\cos\alpha(t)(d\lambda_n/dt-dq/dt)/2(\mu_n)^2\} \\ & - \sin\mu_n(x-a)\{(d\lambda_n/dt-dq/dt)[(x-a)\sin\alpha(t)/2\mu_n + \cos\alpha(t)/2(\mu_n)^3] \\ & + (d\alpha/dt)\sin\alpha(t)/\mu_n\}\end{aligned}$$

Together with Theorem 3.3 the result follows immediately □ .

Theorem 3.5. *Under the same conditions as described in Theorem* 3.4, *we have for any* $t, s, \in J$

$$\sum_{n=0}^{\infty} \left| \frac{y_n(x,t)}{\lambda_n(t)} - \frac{y_n(x,s)}{\lambda_n(s)} \right|^2 < K|t-s|$$

where K *is independent of* x, s *and* t .

Proof. Our eigenvalues are now all positive because of (3.3), and for n sufficiently large, say $n \geq n_0$ the estimate (3.10) holds. Let $\lambda_n(\min)$ be the number defined in (3.10) for $n \geq n_0$, and for $0 \leq n < n_0$ define it to be $\min\{\lambda_n(t) | t \in J\} > 0$.

Set $D = |y_n(x,t)/\lambda_n(t) - y_n(x,s)/\lambda_n(s)|$ and $|y_n(x,t)| \leq S$, (see (3.8)). We then have

$$D^2 \leq 2S(\lambda_n(\min))^{-1}\{(\lambda_n(\min))^{-1}|y_n(x,t) - y_n(x,s)| +$$

$$+ S(\lambda_n(\min))^{-2}|\lambda_n(s) - \lambda_n(t)|\} .$$

Since $\Sigma\, \lambda_n(\min)^{-1} < \infty$, the conclusion follows from the mean value theorem and Theorems 3.3 and 3.4 □ .

We have mentioned before that the boundary conditions that appear in the definition of $A(t)$, (3.2), are a special case of (3.12), (3.13). Therefore the results obtained in Theorems 3.3, 3.4, 3.5 apply to the eigenvalues $\lambda_n(t)$ and eigenfunctions $y_n(x,t)$ of $A(t)$. We need these results to establish the Höldercontinuity in

Theorem 3.6. *For any given* $\rho \in L^2(J,H)$ *there exist positive constants* K_2 *and* β , *such that for any* $s,t \in J$

$$\| B_\rho(t)A^{-1}(t) - B_\rho(s)A^{-1}(s)\| \leq K_2|t-s|^\beta$$

where we have set $B(t,\zeta(\rho)) = B_\rho(t)$.

Proof. In the following C denotes a generic constant. For $f \in H$ $(A^{-1}(t)f)(x) = \Sigma(f_n(t)/\lambda_n(t))y_n(x,t)$, where $f_n(t)$ denotes the n-th Fourier coefficient of f for $t \in J$. Thus if $\|\cdot\|$ denotes the norm in H

$$N = \|B_\rho(t)A^{-1}(t)f - B_\rho(s)A^{-1}(s)f\| \le \|(B_\rho(t) - B_\rho(s))A^{-1}(t)f\| +$$

$$+ \|B_\rho(s)(A^{-1}(t)f - A^{-1}(s)f)\| \equiv I + II .$$

$B_\rho(t)$ is represented by a function $p(t,\zeta(\rho))$ in $L^2(a,b)$. We abbreviate it to $p_\rho(x,t)$. Thus

$$I^2 = \int_a^b |(p_\rho(x,t) - p_\rho(x,s))\Sigma(f_n(t)/\lambda_n(t))y_n(x,t)|^2 dx$$

$$\le S^2\|f\|^2\Sigma(\lambda_n(t))^{-2}\int_a^b |p_\rho(x,t) - p_\rho(x,s)|^2 dx$$

$$\le S^2\|f\|^2 K_\rho^2 \Sigma(\lambda_n(t))^{-2}|t - s|^{2\beta}$$

$$\le \|f\|^2 C|t - s|^{2\beta} ,$$

using Hölder's inequality and because of (3.8), (3.10) and the assumption (iii) on $p_\rho(x,t)$.

$$II^2 = \int_a^b |p_\rho(x,s)\{\Sigma(f_n(t)y_n(x,t)/\lambda_n(t)) - (f_n(s)y_n(x,s)/\lambda_n(s))\}|^2 dx$$

$$\le \int_a^b |p_\rho(x,s)|^2\{\Sigma|f_n(t)||y_n(x,t)/\lambda_n(t) - y_n(x,s)/\lambda_n(s)| +$$

$$+ \Sigma|f_n(t) - f_n(s)||y_n(x,s)/\lambda_n(s)|\}^2 dx$$

$$\leq 2\int_a^b |p_\rho(x,s)|^2\{[\Sigma|f_n(t)||y_n(x,t)/\lambda_n(t) - y_n(x,s)/\lambda_n(s)|]^2 +$$

$$+ [\Sigma|f_n(t) - f_n(s)||y_n(x,s)/\lambda_n(s)|]^2\}dx$$

$$\leq C\|f\|^2|t - s| + 2\int_a^b |p_\rho(x,s)|^2\{\Sigma|f_n(t) - f_n(s)||y_n(x,s)/\lambda_n(s)|\}^2 dx .$$

Again Hölder's inequality has been employed, along with Theorem 3.5 and assumption (ii) on $p_\rho(x,t)$.

The infinite sum in the last integral has to be handled rather carefully. We apply Hölder's inequality to the following form of it.

$$\{\Sigma(|f_n(t) - f_n(s)|/(\lambda_n(s))^{2/3})(|y_n(x,s)|/(\lambda_n(s))^{1/3})\}^2$$

$$\leq \Sigma S^2/(\lambda_n(s))^{2/3} \cdot \Sigma|f_n(t) - f_n(s)|^2/(\lambda_n(s))^{4/3} .$$

The first sum in the last expression is finite since $\lambda_n(s)$ is of the order n^2 . Also

$$|f_n(t) - f_n(s)|^2 \leq \{\int_a^b |f(x)[y_n(x,t) - y_n(x,s)]|dx\}^2$$

$$\leq \|f\|^2\int_a^b |y_n(x,t) - y_n(x,s)|^2 dx .$$

Therefore for any $n \geq 0$,

$$(\lambda_n(s))^{-1/2}|f_n(t) - f_n(s)|^2 \leq \|f\|^2\int_a^b |y_n(x,t) - y_n(x,s)|^2/(\lambda_n(s))^{1/2}dx$$

$$\leq \|f\|^2 \cdot 2S(b-a)C|t - s| ,$$

by the mean value theorem and Theorem 3.4, and thus

$$\Sigma|f_n(t) - f_n(s)|^2/(\lambda_n(s))^{4/3} \leq \|f\|^2 C|t - s|\Sigma(\lambda_n(s))^{-5/6} \leq \|f\|^2 C|t - s| .$$

Hence by assumption (ii) on $p_\rho(x,t)$, $II^2 \le C\|f\|^2|t-s|$. Finally we obtain $N \le I + II \le C\|f\|\,|t-s|^\gamma$, where $\gamma = \min\{\beta, 1/2\}$. This completes the proof of the Höldercontinuity, and with it the proof of Theorem 3.1 □

With similar arguments one can show that for any s, t $\in$ J

$$\|A^{-1}(t) - A^{-1}(s)\| \le K|t-s|^{1/2} .$$

CHAPTER III

0. INTRODUCTION

In this chapter we establish the existence of solutions in a weak sense of the second order semi-linear equation of evolution

$$x'' = (A + B(t,x,x'))x - f(t,x,x') , \tag{0.1}$$

$t \in J = [0,T]$, ($' = d/dt$) , satisfying the boundary conditions

$$L_i(x) = \alpha_{i1}x(0) + \alpha_{i2}x'(0) + \beta_{i1}x(T) + \beta_{i2}x'(T) = 0 , \quad i = 1,2 \tag{0.2}$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$. The operator A is assumed to generate a semi-group of compact type on a separable Hilbert space H . The perturbation $B(t,x,y)$ is assumed to be a bounded linear operator in H and $f(t,x,y)$ a function with values in H satisfying a Caratheodory condition and having sublinear growth.

The chapter is divided into four sections. We are mainly concerned with the perturbed equation

$$x'' = (A + B(t))x . \tag{0.3}$$

One can therefore add a positive constant ℓI to A and subtract it again from $B(t)$. In choosing ℓ sufficiently large, we can ensure the existence of a unique weak solution of the boundary value problem

$$x'' = (A + \ell I)x - f(t)$$

and (0,2), provided the boundary conditions are regular. Moreover, the integral operator associated with this problem is compact. This will be dealt with in Section 1.

It is well known, Becker [6], that the set S of bounded, strongly measurable operator functions $B(t)$ such that $\|B(t) - \ell I\| \leq P$ for almost every $t \in J$, is compact in the weak operator topology in $L^2(J,H)$. If we assume that for $B(t)$ in S the homogeneous equation (0.3) with the boundary conditions (0.2) has only the zero solution, then there exists a unique weak solution to the boundary value problem

$$x'' = (A + B(t))x - f(t)$$

and (0.2). The associated integral operator in $L^2(J,H)$ is compact and bounded uniformly for $B(t)$ in S. This is shown in Section 2.

The results mentioned guarantee a unique solution to the linearized version of the boundary value problem (0.1), (0.2), (see (3.12)), provided $B(t,\psi,\psi')$ belongs to S and f satisfies the above mentioned conditions. This, together with Schauder's fixed point theorem, will imply the existence of a weak solution of the boundary value problem (0.1), (0.2). The necessary convergence result is based on the weak compactness of S and on the compactness of certain integral operators. This forms the content of Theorem 3.1. We deal with this whole situation in Section 3.

In Section 4 an application of the theory will be given, in which A represents an elliptic boundary value problem. The uniqueness assumption mentioned above will be fulfilled if the norm of $B(t) - \ell I$ is sufficiently small (it may increase with ℓ, see (4.4)). We also give an example of a non-differential operator A and describe an integral operator which satisfies the conditions imposed on the perturbation $B(t,x,y)$.

INDEX OF SELECTED DEFINITIONS

$\mathcal{D}(A)$	denotes the domain of A
$V(t)$, $V_\ell(t)$	see (1.3), (1.5)
L_{ij}	see (1.27)
$D(A)$	see (1.28)
d_{ij}	see statements preceding Definition 1.2
V_0	see statements after (1.35)
$L_i(V_0)_t$	see (1.36)
$Q_i(\sigma)$	see (1.39)
$G_\ell(t,\sigma)$	see (1.42)
F	see (2.4)
$G_B(t,\sigma)$	see (2.10)
$\mathcal{S}$	see (2.17)
$\mathcal{H}$, $\mathcal{H}^{1,2}$	see (3.6)

1. THE GREEN'S FUNCTION FOR THE UNPERTURBED EQUATION

In this section we consider the second order evolution equation

$$x'' = (A + \ell I)x - f(t) \tag{1.1}$$

($' = d/dt$) , $t \in J = [0,T]$, in a separable Hilbert space H . Together with eqn. (1.1) we are concerned with the system of boundary conditions

$$L_i(x) = \alpha_{i1}x(0) + \alpha_{i2}x'(0) + \beta_{i1}x(T) + \beta_{i2}x'(T) = 0 \tag{1.2}$$

$i = 1,2$, where $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$. The forms $L_1(u)$, $L_2(u)$ are required to be linearly independent. The linear operator A in H is assumed to satisfy (AIII.1), (AIII.2) below, $\ell \geq 0$ and $f(t)$ belongs to $L^2(J,H)$.

It will be shown in Theorem 1.5 that there exists a unique weak solution (Definition 1.1) of eqn. (1.1) which satisfies the boundary conditions (1.2), provided that they are regular (Definition 1.2) and $\ell \geq 0$ is sufficiently large.

In this section we will follow Krein [43; p.249-269]. However since we are working in $L^2(J,H)$ the notion of a weak solution will be introduced. It is believed to be new and is based on Ball [5].

Assumption (AIII.2) ensures that the integral operator in Theorem 1.6 is compact in $L^2(J,H)$.

Throughout we will suppose that the operator A satisfies

(AIII.1) A is a closed, densely defined linear operator on a separable Hilbert space H with the property that for all $\lambda \geq 0$

$$\|(A + \lambda I)^{-1}\| \leq M/1 + \lambda \ .$$

(AIII.2) A^{-1} is a completely continuous linear operator.

In the following remark we will draw some well known conclusions from (AIII.1) and (AIII.2) which we will need in the sequel.

Remark. Fractional powers of A can be defined, Krein [43; p110]. Moreover the closed linear operator

$$-A^{\frac{1}{2}} \text{ generates a (strongly continuous) analytic semigroup } V(t) . \quad (1.3)$$

Since A^{-1} is compact so is $A^{-\frac{1}{2}}$. Furthermore writing $V(t) = A^{\frac{1}{2}}V(t)A^{-\frac{1}{2}}$ and observing that for $t > 0$ $A^{\frac{1}{2}}V(t)$ is bounded one sees that

$$V(t) \text{ is compact for } t > 0 . \quad (1.4)$$

We refer to [43; p119,264], see also Balakrishnan [4].

Let $\ell \geq 0$. The operator $A + \ell I$ satisfies (AIII.1) and, using the resolvent equation, also (AIII.2). Therefore

$$-(A + \ell I)^{\frac{1}{2}} \text{ generates an analytic semigroup } V_\ell(t) . \quad (1.5)$$

In the following we will abbreviate $A + \ell I$ to A_ℓ or $A + \ell$.

$$(A_\ell)^{-\frac{1}{2}} \text{ is again compact and with it } V_\ell(t) , t > 0 . \quad (1.6)$$

The following facts will be proved in the appendix

$$\|V_\ell(t)\| \leq C , \text{ uniformly for } t \in [0,T] \text{ and } \ell \geq 0 . \quad (1.7)$$

$$\|V_\ell(T)\| \leq C/\sqrt{\ell+1} \quad (1.8)$$

$$\|A_\ell^{-\frac{1}{2}}\| \leq M/\sqrt{\ell+1} \quad (1.9)$$

Concerning the adjoint A^* of A, we have

$$A^* = A^{\frac{1}{2}*} \cdot A^{\frac{1}{2}*} \tag{1.10}$$

$$z \in \mathcal{D}(A^*) \text{ if and only if } z = (A^{-\frac{1}{2}})^* y \text{ for } y \in \mathcal{D}(A^{\frac{1}{2}*}) , \tag{1.11}$$

where $\mathcal{D}(A^*)$ denotes the domain of A^* .

For the first order equation of evolution $x' = Ax - f(t)$ the definition of a weak solution is usually given as follows, see for example Ball [5]. A function $u(t) \in C(J,H)$ is a *weak solution* of the above equation if and only if for every $v \in \mathcal{D}(A^*)$ the function $(u(t),v)$ is absolutely continuous on J and

$$(u(t),v)' = (u(t),A^*v) - (f(t),v)$$

for almost all $t \in J$.

It is well known, [5], that if A generates a strongly continuous semigroup, the evolution equation has a unique weak solution $u(t)$ with $u(0) = x_0$.

We now introduce the definition of a weak solution for the second order equation

$$x'' = Ax - f(t) , \tag{1.12}$$

$f(t) \in L^2(J,H)$.

<u>Definition 1.1</u>. A function $u(t) \in C(J,H)$ is a *weak solution* of (1.12) if and only if

(α) $u'(t)$ exists on $(0,T)$ and $A^{-\frac{1}{2}}u'(t)$ belongs to $C(J,H)$.

(β) $(u(t),z)'$ is absolutely continuous on J for any $z \in \mathcal{D}(A^*)$.

(γ) For any $z \in \mathcal{D}(A^*)$ and for almost all $t \in J$

$$(u(t),z)'' = (u(t),A^*z) - (f(t),z) . \tag{1.13}$$

Theorem 1.1. *Every weak solution of* (1.1) *is of the form*

$$u(t) = V_\ell(t)x_0 + V_\ell(T-t)y_T$$
$$+\frac{1}{2}\int_0^t V_\ell(t-\sigma)A_\ell^{-\frac{1}{2}}f(\sigma)d\sigma + \frac{1}{2}\int_t^T V_\ell(\sigma-t)A_\ell^{-\frac{1}{2}}f(\sigma)d\sigma \ , \tag{1.14}$$

where x_0 , $y_T \in H$. *Furthermore every function of the form* (1.14) *is a weak solution of* (1.1).

Proof. Suppose $u(t)$ is a given weak solution and define

$$v(t) = A_\ell^{-\frac{1}{2}}u'(t) \ .$$

We then have

$$(u(t),z)' = (v(t),A_\ell^{\frac{1}{2}*}z) \tag{1.15}$$

for almost all $t \in J$ and any $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$, and also

$$(v(t),z)' = (u(t),A_\ell^{\frac{1}{2}*}z) - (f(t),A_\ell^{-\frac{1}{2}*}z) \tag{1.16}$$

again for almost all $t \in J$ and any $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$. Now we substitute

$$x = (1/2)(u-v) \quad \text{and} \quad y = (1/2)(u+v)$$

and obtain from (1.15), (1.16) for almost all $t \in J$ and any $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$

$$(x(t),z)' = -(x(t),A_\ell^{\frac{1}{2}*}z) + (1/2)(f(t),A_\ell^{-\frac{1}{2}*}z) \tag{1.17}$$

and

$$(y(t),z)' = (y(t),A_\ell^{\frac{1}{2}*}z) - (1/2)(f(t),A_\ell^{-\frac{1}{2}*}z) \ . \tag{1.18}$$

Both $x(t)$ and $y(t)$ belong to $C(J,H)$, since $u(t)$ and $v(t)$ do. In addition $(x(t),z)$ and $(y(t),z)$ are absolutely continuous for $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$, because

$$(u(t),z)' = (A_\ell^{-\frac{1}{2}}u'(t),A_\ell^{\frac{1}{2}*}z)$$

is continuous for $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$ and

$$(v(t),z) = (u(t),A_\ell^{-\frac{1}{2}*}z)'$$

is absolutely continuous by definition for all $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$.

Since $x(t)$, $y(t)$ satisfy (1.17), (1.18) respectively, it follows that $x(t)$ is a weak solution of

$$x' = -A_\ell^{\frac{1}{2}}x + 2^{-1}A_\ell^{-\frac{1}{2}}f(t) . \tag{1.19}$$

But for an initial value $x_0 \in H$ the equation has a unique weak solution, namely

$$x(t) = V_\ell(t)x_0 + 2^{-1}\int_0^t V_\ell(t-\sigma)A_\ell^{-\frac{1}{2}}f(\sigma)d\sigma . \tag{1.20}$$

It also follows that $y(t)$ is a weak solution of

$$y' = A_\ell^{\frac{1}{2}}y - 2^{-1}A_\ell^{-\frac{1}{2}}f(t) . \tag{1.21}$$

If we introduce the new variable $\tau = T-t$ and write $y(t) = y(T-\tau) = x(\tau)$, then for the function $x(\tau)$ we arrive at the ordinary Cauchy problem, i.e.

$$\begin{aligned}\frac{d}{d\tau}(x(\tau),z) &= -\frac{d}{dt}(y(t),z) = -(y(t),A_\ell^{\frac{1}{2}*}z) + 2^{-1}(f(t),A_\ell^{-\frac{1}{2}*}z)\\ &= -(x(\tau),A_\ell^{\frac{1}{2}*}z) + 2^{-1}(f(T-\tau),A_\ell^{-\frac{1}{2}*}z) .\end{aligned}$$

This equation has a unique weak solution for a given initial value $x_1 \in H$, namely

$$x(\tau) = V_\ell(\tau)x_1 + 2^{-1}\int_0^\tau V_\ell(\tau-\sigma)A_\ell^{-\frac{1}{2}}f(T-\sigma)d\sigma .$$

Consequently equation (1.21) has a unique weak solution for a given *end-value* $y_T \in H$, namely

$$y(t) = V_\ell(T-t)y_T + 2^{-1}\int_t^T V_\ell(\sigma-t)A_\ell^{-\frac{1}{2}}f(\sigma)d\sigma \ . \tag{1.22}$$

Hence if $u(t)$ is a weak solution of (1.1) it will be of the form $x(t) + y(t)$, where $x(t)$, $y(t)$ are given by (1.20), (1.22) respectively.

We will now show that a function of the form (1.14) is a weak solution of (1.1). For $t \in (0,T)$ one obtains

$$\begin{aligned} u'(t) &= -A_\ell^{\frac{1}{2}}V_\ell(t)x_0 + A_\ell^{\frac{1}{2}}V_\ell(T-t)y_T \\ &\quad - 2^{-1}\int_0^t V_\ell(t-\sigma)f(\sigma)d\sigma + 2^{-1}\int_t^T V_\ell(\sigma-t)f(\sigma)d\sigma \ , \end{aligned} \tag{1.23}$$

which exists since $V_\ell(t)$ is an analytic semigroup. It follows now from the strong continuity of $V_\ell(t)$, the dominated convergence theorem and the method used in the proof of I Theorem 3.1 and elsewhere that $u(t)$ and $A_\ell^{-\frac{1}{2}}u'(t)$ belong to $C(J,H)$. We note that $u'(t)$ belongs to $C(J,H)$ if and only if x_0, $y_T \in \mathcal{D}(A_\ell^{\frac{1}{2}})$.

Concerning equation (1.13) consider

$$\begin{aligned} (u(t),z)'' &= \frac{d}{dt}\{(-A_\ell^{-1}A_\ell^{\frac{1}{2}}V_\ell(t)x_0 + A_\ell^{-1}A_\ell^{\frac{1}{2}}V_\ell(T-t)y_T, A_\ell^* z) \\ &\quad - 2^{-1}\int_0^t (V_\ell(t-\sigma)A_\ell^{-1}f(\sigma), A_\ell^* z)d\sigma + 2^{-1}\int_t^T (V_\ell(\sigma-t)A_\ell^{-1}f(\sigma), A_\ell^* z)d\sigma\} \\ &= (V_\ell(t)x_0 + V_\ell(T-t)y_T, A_\ell^* z) - (f(t),z) \\ &\quad + 2^{-1}\int_0^t (V_\ell(t-\sigma)A_\ell^{-\frac{1}{2}}f(\sigma), A_\ell^* z)d\sigma + 2^{-1}\int_t^T (V_\ell(\sigma-t)A_\ell^{-\frac{1}{2}}f(\sigma), A_\ell^* z)d\sigma \\ &= (u(t), A_\ell^* z) - (f(t),z) \ . \end{aligned}$$

The equation holds for almost all $t \in J$ and any $z \in \mathcal{D}(A_\ell^*)$. It thus remains to show that

$$(u(t),z)' - (u(0),z)' = \int_0^t (u(s),z)''ds$$

for $z \in \mathcal{D}(A_\ell{}^*)$.

Let (f_n) be a sequence in $C(J,H)$ with $f_n \to f$ in $L^2(J,H)$, and replace f by f_n in eqn. (1.1). It then follows that for any n , $u_n(t)$ given by (1.14) is a weak solution of (1.1), since

$$(u_n(t),z)'' = (u_n(t),A_\ell{}^*z) - (f_n(t),z)$$

is continuous. By (1.14) we obtain immediately that the sequence $(u_n(t))$ converges to $u(t)$ in $C(J,H)$. For any $t \in J$ and $z \in \mathcal{D}(A_\ell{}^{\frac{1}{2}*})$ we also have

$$(u_n(t)-u(t),z)' = 2^{-1}(-\int_0^t V_\ell(t-\sigma)A_\ell^{-\frac{1}{2}}[f_n(\sigma) - f(\sigma)]d\sigma +$$
$$+ \int_t^T V_\ell(\sigma-t)A_\ell^{-\frac{1}{2}}[f_n(\sigma)-f(\sigma)]d\sigma, A_\ell{}^{\frac{1}{2}*}z) \quad .$$

The two integrals in the last expression converge strongly to zero, uniformly in t . Therefore $(u_n(t),z)'$ converges to $(u(t),z)'$, $(n \to \infty)$ for any $t \in J$ and $z \in \mathcal{D}(A_\ell{}^{\frac{1}{2}*})$. Now consider

$$(u_n(t),z)' - (u_n(0),z)' = \int_0^t (u_n(\sigma),A_\ell{}^*z) - (f_n(\sigma),z)d\sigma \ , \ z \in \mathcal{D}(A_\ell{}^*) \ .$$

The left hand side of the above equation approaches

$$(u(t),z)' - (u(0),z)'$$

as $n \to \infty$, while the right hand side tends to

$$\int_0^t (u(\sigma),A_\ell{}^*z) - (f(\sigma),z)d\sigma \ .$$

But the limit is unique. Therefore the proof is complete □ .

Now the boundary conditions (1.2) are taken into account. Since for a weak solution $u(t)$, $u'(0)$ and $u'(T)$ may not be defined, we first consider

$$A_\ell^{-\frac{1}{2}} L_i(u) = 0 \quad (i = 1,2) , \tag{1.24}$$

in connection with the homogeneous equation

$$x'' = (A + \ell I)x . \tag{1.25}$$

We are looking for conditions under which the boundary value problem (1.24), (1.25) has only the zero solution.

<u>Remark</u>. A function $u(t)$ satisfying $(L_i(u),z) = 0$ $(i = 1,2)$ for all $z \in \mathcal{D}(A_\ell^{\frac{1}{2}*})$ also satisfies (1.24) .

Substituting (1.14) (for $f \equiv 0$) into (1.24) to solve for x_0 and y_T yields

$$\begin{aligned} L_{11}\, x_0 + L_{12}\, y_T &= 0 \\ L_{21}\, x_0 + L_{22}\, y_T &= 0 \end{aligned} \tag{1.26}$$

where we have set for $i = 1,2$

$$L_{i1} = \alpha_{i1} A_\ell^{-\frac{1}{2}} - \alpha_{i2} I + \beta_{i1} A_\ell^{-\frac{1}{2}} V_\ell(T) - \beta_{i2} V_\ell(T)$$

and (1.27)

$$L_{i2} = \alpha_{i1} A_\ell^{-\frac{1}{2}} V_\ell(T) + \alpha_{i2} V_\ell(T) + \beta_{i1} A_\ell^{-\frac{1}{2}} + \beta_{i2} I .$$

All the operators entering into this system are linear combinations of the bounded operators I, $A_\ell^{-\frac{1}{2}}$, $V_\ell(T)$ and $A_\ell^{-\frac{1}{2}} V_\ell(T)$, commuting with one another. Therefore the system may be solved just as in the scalar case. Just as in [43; p252] an important role is played by the operator determinant of (1.26), i.e.

$$D(A_\ell) = \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} \tag{1.28}$$

Solving the system (1.26) we arrive at the equations

$$D(A_\ell)x_0 = 0 \quad \text{and} \quad D(A_\ell)y_T = 0 \quad .$$

If the operator determinant is invertible then clearly x_0 , y_T are zero and thus zero is the only solution of (1.24), (1.25). The converse is also true.

Theorem 1.2. $D(A_\ell)$ *is invertible if and only if the zero solution is the only weak solution of* (1.25) *which satisfies* (1.24).

The proof is the same as in Krein [43; p254] with only minor modifications □ .

Corollary 1.3. *If* $D(A_\ell)$ *is invertible then zero is the only weak solution of* (1.25) *which satisfies* (1.2).

Proof. If v is a nonzero weak solution of (1.25) that satisfies (1.2) then $L_i(v) = 0$, $(i = 1,2)$. Therefore (1.24) is also fulfilled, and $D(A_\ell)$ is not invertible.

The converse need not hold, since if a nonzero weak solution of (1.25) satisfies (1.24), the equations in (1.2) may not vanish. Indeed, they may not even be defined □ .

If we require the coefficients α_{ij}, β_{ij} of (1.2) to satisfy certain regularity conditions and if $\ell \geq 0$ is sufficiently large, then we can show that the operator determinant $D(A_\ell)$ is invertible. Therefore we introduce the following definition which is due to

M.A. Naimark, see [43; p259].

Denote with d_{ij} $(i < j)$ the minor of the matrix

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \end{bmatrix}$$

of the coefficients of the boundary conditions (1.2), composed of the i-th and j-th columns.

Definition 1.2. The boundary conditions (1.2) will be called *regular* if one of the following conditions holds.

(a) $d_{24} \neq 0$

(b) $d_{24} = 0$, but $|\alpha_{12}| + |\beta_{12}| > 0$ and $d_{23} - d_{14} \neq 0$

(c) $\alpha_{12} = \beta_{12} = \alpha_{22} = \beta_{22} = 0$ but $d_{13} \neq 0$.

Theorem 1.4. *If the boundary conditions* (1.2) *are regular and if* $\ell \geq 0$ *is sufficiently large then the operator determinant* $D(A_\ell)$ *is invertible. Equation* (1.25) *together with the boundary conditions* (1.2) *therefore admit only the zero solution.*

Proof. Upon multiplying out $D(A_\ell)$ of (1.28) one obtains

$$\begin{aligned} D(A_\ell) = & -d_{24}I + (d_{14}-d_{23})A_\ell^{-\frac{1}{2}} + d_{13}A_\ell^{-1} + 2(d_{12}+d_{34})A_\ell^{-\frac{1}{2}}V_\ell(T) \\ & + [d_{24}I + (d_{14}-d_{23})A_\ell^{-\frac{1}{2}} - d_{13}A_\ell^{-1}]V_\ell(2T) \ . \end{aligned} \tag{1.29}$$

In case (a) $D(A_\ell)$ can be written in the form

$$D(A_\ell) = c_1(I-R_{\ell 1}) \tag{1.30}$$

where c_1 is a suitable constant and $R_{\ell 1}$ is some polynomial in the bounded linear operators $A_\ell^{-\frac{1}{2}}$ and $V_\ell(T)$ (without a constant term).

In case (b) the coefficient matrix can be brought by Gauss-reduction into the form

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha'_{21} & 0 & \beta'_{21} & 0 \end{bmatrix} \tag{1.31}$$

and $D(A_\ell)$ can be represented as

$$D(A_\ell) = c_2 A_\ell^{-\frac{1}{2}}(I-R_{\ell 2}) \tag{1.32}$$

where again c_2 is a suitable constant and $R_{\ell 2}$ some polynomial in $A_\ell^{-\frac{1}{2}}$ and $V_\ell(T)$ without a constant term.

In case (c), after Gauss-reduction, the coefficient matrix becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.33}$$

and $D(A_\ell)$ can be written as

$$D(A_\ell) = c_3 A_\ell^{-1}(I-R_{\ell 3}) \tag{1.34}$$

where c_3 is a suitable constant and $R_{\ell 3}$ some polynomial in $V_\ell(T)$ without a constant term.

Because of (1.8), (1.9) $\ell \geq 0$ can be chosen so large that $\|R_{\ell m}\| < 1$ for $m = 1,2,3$. This implies that 1 belongs to the resolvent set of $R_{\ell m}$ for $m = 1,2,3$. Therefore $D(A_\ell)$ is invertible.

We note that in cases (b) and (c), $D(A_\ell)^{-1}$ is unbounded and by virtue of (1.6) the operators $R_{\ell m}$ are compact $\square$.

We now return to equation (1.1), with $f \in L^2(J,H)$ and seek to determine uniquely the elements x_0, $y_T \in H$ in (1.14), to ensure a unique weak solution of the boundary value problem (1.1), (1.2).

Definition 1.3. An operator-function $G(t,s)$ such that

$$\int_0^T G(t,s) \cdot ds$$

is a bounded linear operator in $L^2(J,H)$ is called the *Green's function* for the boundary value problem (1.1), (1.2) if and only if for every $f \in L^2(J,H)$

$$u(t) = \int_0^T G(t,s)f(s)ds$$

is a weak solution of (1.1) that satisfies (1.2).

Theorem 1.5. *Assume that the boundary conditions are regular and that* $\ell \geq 0$ *is sufficiently large. Then the boundary value problem* (1.1), (1.2) *has a unique weak solution for every* $f \in L^2(J,H)$. *It is given by*

$$u(t) = \int_0^T G_\ell(t,\sigma)f(\sigma)d\sigma$$

where $G_\ell(t,\sigma)$ *is the Green's function of equation* (1.42).

Proof. *Uniqueness*. Suppose $u(t)$ and $v(t)$ are both weak solutions of (1.1) satisfying (1.2). It follows that $u(t) - v(t)$ is a weak solution of (1.25) which fulfils (1.2). Since $D(A_\ell)$ is invertible, by Corollary 1.3, $u(t) = v(t)$.

Existence. A weak solution $u(t)$ of (1.1) will be of the form (1.14). Upon substituting it into (1.24) one obtains for $i = 1,2$

$$\begin{aligned} A_\ell^{-\frac{1}{2}}L_i(u) &= L_{i1}x_0 + L_{i2}y_T + \\ &+ 2^{-1}A_\ell^{-\frac{1}{2}}\int_0^T [\alpha_{i1}V_\ell(\sigma)A_\ell^{-\frac{1}{2}} + \alpha_{i2}V_\ell(\sigma) + \beta_{i1}V_\ell(T-\sigma)A_\ell^{-\frac{1}{2}} - \beta_{i2}V_\ell(T-\sigma)]f(\sigma)d\sigma = 0. \end{aligned} \tag{1.35}$$

Introducing

$$V_0(t,\sigma) = \begin{cases} 2^{-1}V_\ell(t-\sigma)A_\ell^{-\frac{1}{2}} & \text{for} \quad t \geq \sigma \\ 2^{-1}V_\ell(\sigma-t)A_\ell^{-\frac{1}{2}} & \text{for} \quad t \leq \sigma \end{cases}$$

one has

$$(V_0)_{t=0} = 2^{-1}V_\ell(\sigma)A_\ell^{-\frac{1}{2}} , \quad (V_0)_{t=T} = 2^{-1}V_\ell(T-\sigma)A_\ell^{-\frac{1}{2}}$$

and

$$(\partial V_0/\partial t)_{t=0} = 2^{-1}V_\ell(\sigma) , \quad (\partial V_0/\partial t)_{t=T} = -2^{-1}V_\ell(T-\sigma)$$

and thus we set

$$L_i(V_0)_t = 2^{-1}[\alpha_{i1}V_\ell(\sigma)A_\ell^{-\frac{1}{2}} + \alpha_{i2}V_\ell(\sigma) + \beta_{i1}V_\ell(T-\sigma)A_\ell^{-\frac{1}{2}} - \beta_{i2}V_\ell(T-\sigma)] . \quad (1.36)$$

Therefore the equations become for $i = 1,2$

$$L_{i1}x_0 + L_{i2}y_T = -A_\ell^{-\frac{1}{2}}\int_0^T L_i(V_0)_t f(\sigma)d\sigma = -a_i$$

and we obtain

$$\begin{aligned} D(A_\ell)x_0 &= -L_{22}a_1 + L_{12}a_2 \\ D(A_\ell)y_T &= -L_{11}a_2 + L_{21}a_1 . \end{aligned} \quad (1.37)$$

Since $D(A_\ell)$ is invertible one has for x_0 and y_T

$$\begin{aligned} x_0 &= \int_0^T D(A_\ell)^{-1}A_\ell^{-\frac{1}{2}}[-L_{22}L_1(V_0)_t + L_{12}L_2(V_0)_t]f(\sigma)d\sigma \\ y_T &= \int_0^T D(A_\ell)^{-1}A_\ell^{-\frac{1}{2}}[-L_{11}L_2(V_0)_t + L_{21}L_1(V_0)_t]f(\sigma)d\sigma . \end{aligned} \quad (1.38)$$

We will show that the operator functions in the above integrals

$$\begin{aligned} Q_1(\sigma) &= D(A_\ell)^{-1}[-L_{22}L_1(V_0)_t + L_{12}L_2(V_0)_t] \\ -Q_2(\sigma) &= D(A_\ell)^{-1}[L_{11}L_2(V_0)_t - L_{21}L_1(V_0)_t] \end{aligned} \quad (1.39)$$

are linear operators uniformly bounded in σ. It will then follow, using commutativity and [17; Vol.I,p153] that both x_0 and y_T belong to the domain of $A_\ell^{\frac{1}{2}}$.

In case (a) $D(A_\ell)^{-1}$ is bounded and the result follows from (1.27), (1.30), (1.36).

In case (b) $D(A_\ell)^{-1}$ contains the unbounded factor $A_\ell^{\frac{1}{2}}$. Using (1.31) one sees that L_{21}, L_{22}, $L_2(V_0)_t$ all have the factor $A_\ell^{-\frac{1}{2}}$ and thus the operators are again uniformly bounded by (1.27), (1.32), (1.36).

In case (c) $D(A_\ell)^{-1}$ has the unbounded factor A_ℓ. By (1.33) it follows that L_{11}, L_{12}, L_{21}, L_{22}, $L_1(V_0)_t$ and $L_2(V_0)_t$ all contain $A_\ell^{-\frac{1}{2}}$ as a factor and thus by (1.34) the operators are uniformly bounded.

Since x_0 and y_T belong to $\mathcal{D}(A_\ell^{\frac{1}{2}})$, $u'(t) \in C(J,H)$ as we have noted in the proof of Theorem 1.1. Furthermore $u(t)$ does not only satisfy (1.24) but also (1.2) as can be seen from (1.35).

Applying the operator $D(A_\ell)$ to $u(t)$ and using (1.37) yields

$$
\begin{aligned}
D(A_\ell)u(t) &= V_\ell(t)[-L_{22}a_1 + L_{12}a_2] + V_\ell(T-t)[-L_{11}a_2 + L_{21}a_1] \\
&\quad + D(A_\ell)\int_0^T V_0(t,\sigma)f(\sigma)d\sigma \qquad (1.40) \\
&= \int_0^T G_0(t,\sigma)f(\sigma)d\sigma
\end{aligned}
$$

where, using (1.38), we have set

$$
\begin{aligned}
G_0(t,\sigma) &= V_\ell(t)A_\ell^{-\frac{1}{2}}[L_{12}L_2(V_0)_t - L_{22}L_1(V_0)_t] \\
&\quad - V_\ell(T-t)A_\ell^{-\frac{1}{2}}[L_{11}L_2(V_0)_t - L_{21}L_1(V_0)_t] + D(A_\ell)V_0(t,\sigma) .
\end{aligned}
$$

The operator function $G_0(t,\sigma)$ can be written in the form of a determinant, c.f. [43; p268],

$$G_0(t,\sigma) = \begin{vmatrix} V_\ell(t)A_\ell^{-\frac{1}{2}} & V_\ell(T-t)A_\ell^{-\frac{1}{2}} & V_0(t,\sigma) \\ L_{11} & L_{12} & L_1(V_0)_t \\ L_{21} & L_{22} & L_2(V_0)_t \end{vmatrix} \tag{1.41}$$

If we set

$$G_\ell(t,\sigma) = D(A_\ell)^{-1}G_0(t,\sigma) \tag{1.42}$$

we finally arrive at

$$u(t) = \int_0^T G_\ell(t,\sigma)f(\sigma)d\sigma . \tag{1.43}$$

In the light of what we have shown in (1.39), the operator function $G_\ell(t,\sigma)$ is uniformly bounded in all three cases (a), (b), (c).

Since $u(t)$ of eqn. (1.43) can be brought into the form (1.14), (c.f. (1.40)), it follows that it is a weak solution of eqn. (1.1).

It is now verified that $u(t)$ of eqn. (1.43) satisfies the boundary conditions (1.2). It is easy to show that

$$L_i(u) = \int_0^T L_i(G_\ell(t,\sigma))_t f(\sigma)d\sigma$$

where the subscript t in $L_i(G_\ell(t,\sigma))_t$ indicates that the boundary operators L_i act on $G_\ell(t,\sigma)$ as on a function of t. Therefore it remains to verify that

$$L_i(G_\ell(t,\sigma))_t = 0 \tag{1.44}$$

for $i = 1,2$ and $\sigma \in J$.

By (1.39) one obtains for the expression in (1.44),

$$L_i(G_\ell(t,\sigma))_t = L_{i1}Q_1(\sigma) - L_{i2}Q_2(\sigma) + L_i(V_0)_t .$$

Using the commutativity again, one arrives at

$$L_i(G_\ell(t,\sigma))_t = D(A_\ell)^{-1}\{(L_{i1}L_{12} - L_{i2}L_{11})L_2(V_0)_t +$$

$$+ (L_{i2}L_{21} - L_{i1}L_{22})L_1(V_0)_t\} + L_i(V_0)_t = 0 ,$$

thereby completing the proof □ .

The next result in this section deals with the compactness of the integral operator in (1.43).

Theorem 1.6. *The integral operator* $\int_0^T G_\ell(t,\sigma) \cdot d\sigma$ *is compact in* $L^2(J,H)$.

Proof. With (1.39) we can write $G_\ell(t,\sigma)$ in the form

$$G_\ell(t,\sigma) = V_\ell(t)A_\ell^{-\frac{1}{2}}Q_1(\sigma) - V_\ell(T-t)A_\ell^{-\frac{1}{2}}Q_2(\sigma) + V_0(t,\sigma) . \tag{1.45}$$

It follows from what was shown in (1.39) that $Q_i(\sigma)$, $(i = 1,2)$ are uniformly bounded in σ . Further, by (1.6) we conclude that for any $\sigma \in J$, $\sigma \neq t$ and for $t \in (0,T)$, $G_\ell(t,\sigma)$ is a compact operator. Since $G_\ell(t,\sigma)$ is uniformly bounded in t and σ , we can apply I Proposition 1.2 (a). Hence the result follows □ .

Lastly we note that

$$\|G_\ell(t,\sigma)\| \leq C/\sqrt{\ell+1} . \tag{1.46}$$

We use equation (1.45) and suppose of course that ℓ is sufficiently large so that $D(A_\ell)^{-1}$ exists, i.e. $R_{\ell m}$ in (1.30), (1.32), (1.34) satisfy $\|R_{\ell m}\| < 1$. Therefore we obtain $\|(I - R_{\ell m})^{-1}\| \leq (1 - \|R_{\ell m}\|)^{-1} <$ C . By virtue of (1.39) $Q_i(\sigma)$ $(i = 1,2)$ are uniformly bounded in σ and in ℓ . Using (1.7), (1.9) and the equation following (1.35), the result follows.

In the next section we will be concerned with the perturbed equation.

2. THE PERTURBED EQUATION

In this section the Green's function $G_B(t,s)$ is constructed for the perturbed second order evolution equation

$$x'' = (A + B(t))x - f(t) , \tag{2.1}$$

with the boundary conditions (1.2). The operator A is the same as in Section 1 and $f(t)$ again belongs to $L^2(J,H)$. The perturbation operator $B(t)$ belongs to the set S_N of Chapter I, i.e. we suppose

(AIII.3) $B(t)$ denotes a strongly measurable map from $J = [0,T]$ into $\mathcal{B}(H)$, the bounded linear operators in H , i.e. $B(t) \in \mathcal{M}(J,H)$ of I Proposition 1.1. Further $\|B(t)\| \leq N$ for almost every $t \in J$ and some constant $N > 0$.

In the sequel the regularity of the boundary conditions of Definition 1.2 will be assumed.

(AIII.4) The boundary conditions (1.2) are regular (Definition 1.2).

Under an additional regularity assumption, we will show in Theorem 2.2, that the boundary value problem (2.1), (1.2) has a unique weak solution. The weak compactness of the set S in (2.17) will also enable us to show that the operators in Theorem 2.3 are uniformly bounded for $B(t)$ in S .

In constructing the Green's function $G_B(t,\sigma)$ for (2.1), (1.2), we consider

$$x'' = ((A+\ell I) + (B(t)-\ell I))x - f(t) \tag{2.2}$$

and obtain heuristically, c.f. Laptev [46]

$$(G_B - G_\ell)'' = ((A+\ell I) + (B(t)-\ell I))G_B - (A+\ell I)G_\ell$$

$$= (A+\ell I)(G_B - G_\ell) + (B(t)-\ell I)G_B .$$

Hence for $x_0 \in H$

$$G_B(t,\sigma)x_0 - G_\ell(t,\sigma)x_0 = -\int_0^T G_\ell(t,s)(B(s) - \ell I)G_B(s,\sigma)x_0 ds .$$

Theorem 2.1. *Suppose that* (AIII.1) - (AIII.4) *hold and that* $\ell \geq 0$ *is sufficiently large. Assume further that the homogeneous equation*

$$x'' = (A+B(t))x \tag{2.3}$$

has only the weak solution $u(t) \equiv 0$ *which satisfies the boundary conditions* (1.2). *Then the linear operator in* $L^2(J,H)$

$$F = \int_0^T G_\ell(t,\sigma)(B(\sigma)-\ell I) - d\sigma \tag{2.4}$$

is compact and -1 *belongs to the resolvent set of* F .

Proof. Since $B(\sigma) - \ell I$ is uniformly bounded for almost every $\sigma \in J$, it follows from the proof of Theorem 1.6 that the kernel of the operator in (2.4) is compact for almost all (t,σ) . Therefore, using I Proposition 1.2(a), the operator in (2.4) is compact. Hence -1 is either an eigenvalue of F or -1 belongs to the resolvent set of F . Suppose -1 is an eigenvalue of F , i.e. there exists a nonzero function $v(t)$ in $L^2(J,H)$ such that

$$-v(t) = \int_0^T G_\ell(t,\sigma)(B(\sigma)-\ell I)v(\sigma)d\sigma . \tag{2.5}$$

We will show that $v(t)$ is a weak solution of the boundary value problem (2.3), (1.2), thereby obtaining a contradiction with the uniqueness assumption.

It follows from (1.44) that $L_i(v) = 0$, $i = 1,2$. In view of Theorem 2.2, we now show that for $f(t) \in L^2(J,H)$, a function $v(t)$ satisfying

$$v(t) = \int_0^T G_\ell(t,\sigma)[f(\sigma) - (B(\sigma)-\ell I)v(\sigma)]d\sigma \qquad (2.6)$$

is a weak solution of (2.1). For $G_\ell(t,\sigma)$ the form of equation (1.45) will be employed. In the sequel we will use the abbreviation

$$g(\sigma) = f(\sigma) - (B(\sigma)-\ell I)v(\sigma) . \qquad (2.7)$$

It is a given function in $L^2(J,H)$.

(α) $\underline{v(t) \in C(J,H)}$

This can be shown easily by an application of the dominated convergence theorem, since $G_\ell(t,\sigma)$ is uniformly bounded. The term $V_0(t,\sigma)$ (see after eqn. (1.35)) does not present any extra difficulties, but see (β).

(β) $\underline{v'(t) \in C(J,H)}$

For $\sigma \neq t$ $\partial G_\ell(t,\sigma)/\partial t \cdot g(\sigma) =$

$$= [-V_\ell(t)Q_1(\sigma) - V_\ell(T-t)Q_2(\sigma) + 2^{-1}\{\begin{matrix} V_\ell(\sigma-t) & , t < \sigma \\ -V_\ell(t-\sigma) & , t > \sigma \end{matrix}\}]g(\sigma) \qquad (2.8)$$

i.e. for any $t \in J$, $\partial G_\ell(t,\sigma)/\partial t \cdot g(\sigma)$ exists for almost every $\sigma \in J$. Furthermore, using (1.7), (1.39), its norm is bounded by $C\|g(\sigma)\|$ which is integrable. Therefore for any $t \in J$,

$$v'(t) = \int_0^T \frac{\partial}{\partial t} G_\ell(t,\sigma)g(\sigma)d\sigma . \qquad (2.9)$$

Now let $s < t$ and consider

$$v'(t) - v'(s) = \int_0^T (V_\ell(s)-V_\ell(t))Q_1(\sigma)g(\sigma)d\sigma + \int_0^T (V_\ell(T-s)-V_\ell(T-t))Q_2(\sigma)g(\sigma)d\sigma$$

$$+ 2^{-1}\int_0^s (V_\ell(s-\sigma)-V_\ell(t-\sigma))g(\sigma)d\sigma + 2^{-1}\int_s^t -V_\ell(t-\sigma)g(\sigma)d\sigma$$

$$+ 2^{-1}\int_t^T (V_\ell(\sigma-t)-V_\ell(\sigma-s))g(\sigma)d\sigma + 2^{-1}\int_s^t -V_\ell(\sigma-s)g(\sigma)d\sigma$$

$$\equiv I_1 + I_2 + \dots + I_6 .$$

The integrals I_4 and I_6 clearly converge to zero in norm as $|t-s| \to 0$. The dominated convergence theorem will guarantee that I_1 and I_2 will also tend to zero in norm, as $|t-s| \to 0$. Concerning the integrals in I_3 and I_5 we employ the method used in I Theorem 3.1 (in the continuity proof of the fixed point) and apply the dominated convergence theorem. We can thereby show that the norms of I_3 and I_5 approach zero as $|t-s| \to 0$.

(γ) We now turn to part (γ) of Definition 1.1. Let $z \in \mathcal{D}(A^*)$, then by (2.8), (2.9)

$$(v'(t),z)' = \frac{d}{dt}\{\int_0^T (-A_\ell^{-\frac{1}{2}}V_\ell(t)Q_1(\sigma)g(\sigma) - A_\ell^{-\frac{1}{2}}V_\ell(T-t)Q_2(\sigma)g(\sigma), A_\ell^{\frac{1}{2}*}z)d\sigma$$

$$+ 2^{-1}\int_0^t (-A_\ell^{-\frac{1}{2}}V_\ell(t-\sigma)g(\sigma), A_\ell^{\frac{1}{2}*}z)d\sigma + 2^{-1}\int_t^T (A_\ell^{-\frac{1}{2}}V_\ell(\sigma-t)g(\sigma), A_\ell^{\frac{1}{2}*}z)d\sigma\} .$$

Again differentiation can be carried out inside the integral. Hence by (2.7)

$$(v(t),z)'' = \int_0^T (A_\ell^{-\frac{1}{2}}V_\ell(t)Q_1(\sigma)g(\sigma) - A_\ell^{-\frac{1}{2}}V_\ell(T-t)Q_2(\sigma)g(\sigma), A_\ell^{*}z)d\sigma +$$

$$+ 2^{-1}\int_0^t (A_\ell^{-\frac{1}{2}}V_\ell(t-\sigma)g(\sigma)\ ,\ A_\ell{}^*z)d\sigma + 2^{-1}\int_t^T (A_\ell^{-\frac{1}{2}}V_\ell(\sigma-t)g(\sigma)\ ,\ A_\ell{}^*z)d\sigma$$

$$- (f(t) - (B(t) - \ell I)v(t)\ ,\ z)$$

$$= (\int_0^T [V_\ell(t)A_\ell^{-\frac{1}{2}}Q_1(\sigma) - V_\ell(T-t)A_\ell^{-\frac{1}{2}}Q_2(\sigma) + V_0(t,\sigma)]g(\sigma)d\sigma\ ,\ A_\ell{}^*z) +$$

$$+ (v(t)\ ,\ (B(t)-\ell I)^*z) - (f(t)\ ,\ z)\ .$$

Using (2.6), (2.7) and the fact that $(A + B)^* = A^* + B^*$ for a closed operator A and a bounded operator B, we obtain for $z \in \mathcal{D}(A^*)$ and almost every $t \in J$

$$(v(t)\ ,\ z)'' = (v(t)\ ,\ A_\ell{}^*z + (B(t)-\ell I)^*z) - (f(t)\ ,\ z)$$

$$= (v(t)\ ,\ (A+B(t))^*z) - (f(t)\ ,\ z)\ .$$

(δ) In order to verify the absolute continuity of $(v(t)\ ,\ z)'$ it remains to show that for $z \in \mathcal{D}(A^*)$

$$(v(t)\ ,\ z)' - (v(0)\ ,\ z)' = \int_0^t (v(s)\ ,\ z)''ds\ .$$

In so doing we adopt the method used in the proof of Theorem 1.1. The function $g(t)$ of (2.7) is a given function in $L^2(J,H)$. Let (f_n) be a sequence in $C(J,H)$ such that $f_n \to g$ in $L^2(J,H)$. Hence by Theorem 1.5,

$$v_n(t) = \int_0^T G_\ell(t,\sigma)f_n(\sigma)d\sigma$$

is the unique weak solution of the boundary value problem (1.1), (1.2), for any n. The derivative of $v_n(t)$ is given by the analogue of (2.8), (2.9) .

Since $G_\ell(t,\sigma)$ is uniformly bounded, it is evident that $v_n(t) \to v(t)$ in $C(J,H)$. Using (2.8), (2.9) it also follows that $v_n'(t) \to v'(t)$ in $C(J,H)$. Therefore as $n \to \infty$ the left hand side of the equation

$$(v_n'(t), z) - (v_n'(0), z) = \int_0^t (v_n(s), A_\ell^* z) - (f_n(s), z)ds$$

converges to

$$(v'(t), z) - (v'(0), z),$$

while the right hand side tends to

$$\int_0^t (v(s), A_\ell^* z) - (g(s), z)ds$$
$$= \int_0^t (v(s), A_\ell^* z + (B(s)-\ell I)^* z) - (f(s), z)ds.$$

Hence a function $v(t)$ satisfying (2.6) is a weak solution of the boundary value problem (2.1), (1.2). The proof is complete □.

For any $\sigma \in J$ and $x \in H$, $G_\ell(t,\sigma)x$ is an $L^2(J,H)$ function in t. Thus for the Green's function $G_B(t,\sigma)$ we set

$$G_B(t,\sigma)x = (F+I)^{-1} G_\ell(t,\sigma)x . \tag{2.10}$$

It is an $L^2(J,H)$ function in t, for any $\sigma \in J$ and $x \in H$. By virtue of this definition for G_B, we arrive at the main result of this section. It will be needed in the proof of the existence result (Theorem 3.2).

Theorem 2.2. *Assume that the hypotheses of* Theorem 2.1 *hold. Then for any* $f(t) \in L^2(J,H)$

$$u(t) = \int_0^T G_B(t,\sigma)f(\sigma)d\sigma \tag{2.11}$$

is the unique weak solution of the boundary value problem (2.1), (1.2). *The function* $u(t)$ *satisfies*

$$u(t) = \int_0^T G_\ell(t,\sigma)[f(\sigma) - (B(\sigma)-\ell I)u(\sigma)]d\sigma . \tag{2.12}$$

Furthermore, the operator in $L^2(J,H)$

$$\int_0^T G_B(t,\sigma) \cdot d\sigma \tag{2.13}$$

given by (2.16) *is compact.*

Proof. *Uniqueness.* Suppose that both $u(t)$ and $v(t)$ are weak solutions of (2.1), (1.2). Therefore $z(t) = u(t) - v(t)$ is a weak solution of (2.3), which satisfies (1.2). This contradicts the uniqueness assumption and thus $u(t) = v(t)$.

Existence. By the proof of Theorem 2.1 we know that a function $u(t)$ which satisfies (2.12) is a weak solution of (2.1).

Such a function also fulfils the boundary conditions (1.2), by (1.44). It remains to show that $u(t)$ of equation (2.11) does satisfy (2.12).

First, $u(t)$ belongs to $L^2(J,H)$. In fact, for $f \in L^2(J,H)$, $G_B(t,\sigma)f(\sigma)$ is an $L^2(J,H)$ function in t for almost every $\sigma \in J$ and

$$\|G_B(t,\sigma)f(\sigma)\|_{L^2} \le \|(F+I)^{-1}\| \|G_\ell(t,\sigma)f(\sigma)\|_{L^2} .$$

By Hölder's inequality and Fubini's theorem

$$\|u\|^2 \le \int_0^T \{\int_0^T \|G_B(t,\sigma)f(\sigma)\|d\sigma\}^2 dt \le T\int_0^T \{\int_0^T \|G_B(t,\sigma)f(\sigma)\|^2 dt\}d\sigma$$

$$\leq \|(F+I)^{-1}\|^2 T\int_0^T\int_0^T \|G_\ell(t,\sigma)f(\sigma)\|^2 dt d\sigma \ ,$$

which is finite, since $G_\ell(t,\sigma)$ is uniformly bounded.

We can now write

$$x(t) = \int_0^T G_\ell(t,\sigma)f(\sigma)d\sigma = \int_0^T (F+I)G_B(t,\sigma)f(\sigma)d\sigma \tag{2.14}$$

$$= \int_0^T \{\int_0^T G_\ell(t,s)(B(s)-\ell I)G_B(s,\sigma)f(\sigma)ds + G_B(t,\sigma)f(\sigma)\}d\sigma$$

$$= \int_0^T\int_0^T G_\ell(t,s)(B(s)-\ell I)G_B(s,\sigma)f(\sigma)d\sigma ds + \int_0^T G_B(t,\sigma)f(\sigma)d\sigma$$

$$= \int_0^T G_\ell(t,s)(B(s)-\ell I)\{\int_0^T G_B(s,\sigma)f(\sigma)d\sigma\}ds + u(t)$$

$$= (F+I)u(t) \ . \tag{2.15}$$

It follows that $u(t)$ satisfies (2.12). From (2.10), (2.14) and (2.15) we conclude that

$$\int_0^T G_B(t,\sigma)f(\sigma)d\sigma = (F+I)^{-1}\int_0^T G_\ell(t,\sigma)f(\sigma)d\sigma \tag{2.16}$$

$$= \int_0^T (F+I)^{-1}G_\ell(t,\sigma)f(\sigma)d\sigma \ .$$

Hence the linear operator of eqn. (2.13) is bounded. It is also compact, since the integral operator of Theorem 1.6 is compact in $L^2(J,H)$ and $(F+I)^{-1}$ is bounded in $L^2(J,H)$. This completes the proof □ .

As usual we shall assume that $\ell \geq 0$ is sufficiently large so that by Theorem 1.5 the Green's function $G_\ell(t,\sigma)$ of (1.42) exists. Now let

$$S = \{B(t) \in M(J,H) \,|\, \|B(t) - \ell I\| \leq P \quad \text{for a.e.} \quad t \in J\} \tag{2.17}$$

The set S is contained in S_N of I Proposition 1.1, where $N = P + \ell$. It is the translation of the weakly compact set S_P and is itself weakly compact by the continuity of translation, see Becker [6; p44]. By [17; Vol.I,VI 9.2] S is also closed in the weak topology in $B(L^2(J,H))$.

When we view $B(t)$ as an operator in $L^2(J,H)$ we indicate this by writing $\tilde{B}$ instead of $B(t)$. If the uniqueness condition of Theorem 2.1 holds for every $B(t)$ in S then, because of the weak compactness of S and the compactness of the integral operator in Theorem 1.6, the operators in (2.13) are uniformly bounded for $B(t)$ in S. In this way we do not need a priori estimates of the derivatives.

<u>Theorem 2.3</u>. *Assume that the hypotheses of* Theorem 2.1 *are satisfied for every* $B(t) \in S$. *Then the operators in* $B(L^2(J,H))$

$$\{\int_0^T G_\ell(t,\sigma)(B(\sigma)-\ell I) \quad - \quad d\sigma + I\}^{-1}$$

are uniformly bounded for $B(t)$ *in* S. *Therefore by* (2.16) *the compact operators*

$$\int_0^T G_B(t,\sigma) \quad - \quad d\sigma$$

of (2.13) *are uniformly bounded for* $B(t)$ *in* S.

Proof. Suppose to the contrary that $(F+I)^{-1}$ are unbounded in $B(t) \in S$. Thus there exist sequences $(B_n(t)) \subset S$ and (x_n) in $L^2(J,H)$, $\|x_n\| = 1$ such that the sequence in $L^2(J,H)$

$$y_n = \left(\int_0^T G_\ell(t,\sigma)(B_n(\sigma)-\ell I) - d\sigma + I\right)^{-1} x_n$$

diverges. Set $z_n = y_n/\|y_n\|$ and $w_n = x_n/\|y_n\|$, so that we can write

$$w_n(t) = z_n(t) + \int_0^T G_\ell(t,\sigma)(B_n(\sigma)-\ell I) z_n(\sigma) d\sigma . \tag{2.18}$$

Since $(B_n(t)-\ell I)z_n(t)$ is a bounded sequence in $L^2(J,H)$, we can find a subsequence $(B_m(t)-\ell I)z_m(t)$ which converges weakly to $b(t)$ (say). The integral operator of Theorem 1.6 is compact. Hence by (2.18)

$$w_m - z_m \to \int_0^T G_\ell(t,\sigma) b(\sigma) d\sigma = -z , \quad (m\to\infty)$$

strongly in $L^2(J,H)$. Therefore as $m \to \infty$

$$\|z-z_m\| \le \|w_m - z_m + z\| + \|w_m\| \to 0 . \tag{2.19}$$

Since $(\tilde{B}_m)$ is a sequence in a weakly compact set in $\mathcal{B}(L^2(J,H))$, there exists a subsequence $(\tilde{B}_j)$ weakly converging to $\tilde{B}_0 \in S$ (say). We will now show that

$$(\tilde{B}_j - \ell I) z_j \to (\tilde{B}_0 - \ell I) z \tag{2.20}$$

weakly in $L^2(J,H)$. Because of (2.19) it suffices to prove that for any y in $L^2(J,H)$

$$p_j = [\tilde{B}_j z_j - \tilde{B}_0 z , y]$$

tends to zero as $j \to \infty$. We obtain

$$|p_j| \leq |[\tilde{B}_j(z_j-z),y]| + |[(\tilde{B}_j-\tilde{B}_0)z,y]|$$

$$\leq N\|y\|\|z_j-z\| + |[(\tilde{B}_j-\tilde{B}_0)z,y]| \to 0$$

as $j \to \infty$. Therefore by (2.18) and (2.20) one has

$$w_j - z_j \to \int_0^T G_\ell(t,\sigma)(B_0(\sigma)-\ell I)z(\sigma)d\sigma$$

strongly in $L^2(J,H)$. Finally by considering in $L^2(J,H)$

$$\|z + \int_0^T G_\ell(t,\sigma)(B_0(\sigma)-\ell I)z(\sigma)d\sigma\|$$

$$= \|z-z_j+w_j - \int_0^T G_\ell(t,\sigma)(B_j(\sigma)-\ell I)z_j(\sigma)d\sigma +$$

$$+ \int_0^T G_\ell(t,\sigma)(B_0(\sigma)-\ell I)z(\sigma)d\sigma\|$$

$$\leq \|z-z_j\| + \|w_j\| + \|\int_0^T G_\ell(t,\sigma)\{(B_0(\sigma)-\ell I)z(\sigma) - (B_j(\sigma)-\ell I)z_j(\sigma)\}d\sigma\|$$

which tends to zero as $j \to \infty$, we obtain

$$z(t) = -\int_0^T G_\ell(t,\sigma)(B_0(\sigma)-\ell I)z(\sigma)d\sigma .$$

Because $\|z\| = 1$, it follows from the proof of Theorem 2.1 that $z(t)$ is a weak solution of (2.3), (1.2). This contradicts the uniqueness assumption, since $B_0(t)$ belongs to S . This completes the proof $\square$.

We will need these results in the next section, where we introduce nonlinearity.

3. THE NON-LINEAR EQUATION

In this section we shall prove the existence of a weak solution of the equation

$$x'' = (A + B(t,x,x'))x - f(t,x,x') \tag{3.1}$$

which satisfies the regular boundary conditions (1.2).

In a preliminary paragraph a suitable Hilbert space $H^{1,2}$ will be introduced which is contained in $L^2(J,H)$ and in which $f(t) \in L^2(J,H)$ has a distributional derivative.

In another preparatory step we gain a convergence result (Theorem 3.1) which is of the same type as I Theorem 2.7. Namely for weakly convergent sequences $(B_n(t))$ in S of (2.17) and (f_n) in $L^2(J,H)$, we obtain strong convergence in $H^{1,2}$ for the weak solutions $u_n(t)$ of the equations

$$x'' = (A + B_n(t))x - f_n(t) \tag{3.2}$$

which satisfy the boundary conditions (1.2). This refinement from weak to strong convergence is due to the weak compactness of S and the compactness in $L^2(J,H)$ of certain integral operators, see Theorems 1.6 and 2.2.

The main result (Theorem 3.2) will then be proved by the usual linearization trick and an application of Schauder's fixed point theorem.

Notation. The Green's function $G_\ell(t,\sigma)$ of eqn. (1.42) will be abbreviated to $G_A(t,\sigma)$ in this section to avoid confusion. We will use the abbreviation $G_n(t,\sigma)$ for the Green's functions of the perturbed equations (3.2).

In this paragraph we will briefly describe the Hilbert space $H^{1,2}$, which we will abbreviate to H, of functions $f(t) \in L^2(J,H)$ whose first distributional derivative belongs to $L^2(J,H)$, c.f. Lions [50; p7].

Any function $f(t)$ in $L^2(J,H)$ defines a linear function $\tilde{f}$ from C_0^∞ into the separable Hilbert space H by

$$\tilde{f}(\psi) = \int_0^T f(t)\psi(t)dt \ , \tag{3.3}$$

where $C_0^\infty = C^\infty(J,\mathbb{C})$ with compact support in J. The function $\tilde{f}$ is also bounded in the following sense

$$\|\tilde{f}(\psi)\| \leq \int_0^T \|f(t)\| \, |\psi(t)| dt \leq \|f\|_{L^2(J,H)} \|\psi\|_{L^2(J)} \ ,$$

using Hölder's inequality.

By definition $f(t) \in L^2(J,H)$ has a *distributional derivative* $\partial f/\partial t$, if there exists $q(t) \in L^2(J,H)$ such that for all $\psi \in C_0^\infty$

$$\tilde{q}(\psi) = -\int_0^T f(t)\psi'(t)dt \ . \tag{3.4}$$

We define $\partial f/\partial t = q(t)$. By (3.3) we have

$$\widetilde{\frac{\partial f}{\partial t}}(\psi) = -\tilde{f}(\psi') \ .$$

The distributional derivative $\partial f/\partial t \in L^2(J,H)$ is uniquely determined, if it exists, since

$$\int_0^T p(t)\psi(t)dt = 0$$

for every $\psi \in C_0^\infty$ implies $p(t) = 0$ in $L^2(J,H)$. This can be seen as follows. For any $h \in H$, $\overline{(p(t),h)}$ belongs to $L^2(J)$. Hence there exists a sequence (ψ_n) in C_0^∞ such that $\psi_n \to \overline{(p(t),h)}$ in $L^2(J)$. Therefore in $\mathbb{C}$

$$0 = (\int_0^T p(t)\psi_n(t)dt,h) = \int_0^T \psi_n(t)(p(t),h)dt$$

which converges to

$$\int_0^T |(p(t),h)|^2 dt \tag{3.5}$$

as $n \to \infty$, by Hölder's inequality. Thus for every $h \in H$ the expression in (3.5) equals zero, i.e. $(p(t),h) = 0$ for every $t \in J \setminus Z_h$, where Z_h has measure zero. Let (h_m) be dense in H . It follows that $(p(t),h_m) = 0$ for every integer m and every $t \in J \setminus Z$, where Z has measure zero, being a countable union of sets of measure zero. Therefore $p(t) = 0$ in $L^2(J,H)$.

By setting

$$H^{1,2} = \{f \in L^2(J,H) \,|\, f \text{ has a distributional derivative } \partial f/\partial t \text{ in } L^2(J,H)\} \tag{3.6}$$

and

$$[f,g]_H = [f,g]_{L^2(J,H)} + [\partial f/\partial t, \partial g/\partial t]_{L^2(J,H)} \tag{3.7}$$

one obtains a Hilbert space. In the sequel we will abbreviate $\partial f/\partial t$ to f' .

(α) First we note that if $f(t)$ is differentiable and $f'(t)$ belongs to $L^2(J,H)$, then

$$\tilde{f}'(\psi) = \int_0^T f'(t)\psi(t)dt = -\int_0^T f(t)\psi'(t)dt ,$$

and thus $f(t) \in H$.

(β) H *is a Hilbert space.* The equation in (3.7) clearly defines an inner product in H . We will show that H is complete. Let (f_n) be a Cauchy sequence in H , i.e.

$$\|f_n - f_m\|_H^2 = \|f_n - f_m\|^2 + \|f_n' - f_m'\|^2$$

converges to zero as $m, n \to \infty$. Because $L^2(J,H)$ is complete there exist functions f_0, f_1 in $L^2(J,H)$ such that $f_n \to f_0$ and $f_n' \to f_1$ in $L^2(J,H)$. It therefore remains to show that $f_n \to f_0$ in H , i.e. that $f_0 \in H$ and $f_0' = f_1$.

It follows by Hölder's inequality that for any $\psi \in C_0^\infty$

$$-\int_0^T f_n(t)\psi'(t)dt \to -\int_0^T f_0(t)\psi'(t)dt$$

and

$$\int_0^T f_n'(t)\psi(t)dt \to \int_0^T f_1(t)\psi(t)dt$$

in H , as $n \to \infty$. But of course for any ψ and all n we have

$$\tilde{f}_n'(\psi) = \int_0^T f_n'(t)\psi(t)dt = -\int_0^T f_n(t)\psi'(t)dt .$$

Therefore

$$\tilde{f}_1(\psi) = \int_0^T f_1(t)\psi(t)dt = -\int_0^T f_0(t)\psi'(t)dt$$

for any $\psi \in C_0^\infty$ and thus by definition $f_1 = f_0'$. Hence H is a Hilbert space.

Next we prove our convergence result which we have described before.

Theorem 3.1. *Suppose that* (AIII.1) - (AIII.4) *are satisfied and that* $\ell \geq 0$ *is sufficiently large. Further assume that for every* $B(t) \in S$ *of* (2.17) *the homogeneous equation* (2.3) *has only the weak solution* $u(t) \equiv 0$ *which satisfies the boundary conditions* (1.2).

If $(B_n(t))$ *is a sequence in* S *weakly convergent to* $B(t)$ *in* $\mathcal{B}(L^2(J,H))$ *and if* (f_n) *is a sequence in* $L^2(J,H)$ *weakly convergent to* f , *then the weak solutions* $u_n(t)$ *of the equations in* (3.2) *which satisfy the boundary conditions* (1.2) *converge strongly in* $H^{1,2}$ *to* $u(t)$, *the weak solution of the boundary value problem* (2.1), (1.2) .

Proof. As indicated before $G_n(t,s)$ will denote the Green's function for eqn. (3.2), while $G_A(t,s)$ denotes that of eqn. (1.42).

By Theorem 2.3 and (2.11) and since (f_n) is weakly convergent, we have

$$\|u_n\|_{L^2(J,H)} = \left\|\int_0^T G_n(t,\sigma)f_n(\sigma)d\sigma\right\| \leq Q ,$$

i.e. (u_n) is a bounded sequence in $L^2(J,H)$. Since $(B_n(t))$ is in S , it follows that $(g_n) = (f_n(t) - (B_n(t) - \ell I)u_n(t))$ is a bounded sequence in $L^2(J,H)$. Denote by (g_j) a weakly convergent subsequence thereof.

The weak solutions $u_n(t)$ of (3.2), (1.2) also satisfy (2.12), i.e.

$$u_n(t) = \int_0^T G_A(t,\sigma)g_n(\sigma)d\sigma . \tag{3.8}$$

Using (2.8), (2.9), we see that the integral operator associated with $u_n'(t)$ is given by

$$\int_0^T \frac{\partial}{\partial t} G_A(t,\sigma) \cdot d\sigma = -\int_0^T [V_\ell(t)Q_1(\sigma) + V_\ell(T-t)Q_2(\sigma)] \cdot d\sigma$$

$$- 2^{-1}\int_0^t V_\ell(t-\sigma) \cdot d\sigma + 2^{-1}\int_t^T V_\ell(\sigma-t) \cdot d\sigma \,. \tag{3.9}$$

Furthermore by the proof of Theorem 1.6, comparing (2.8) with (1.45), we can conclude that the operator in (3.9) is compact in $L^2(J,H)$.

Therefore it follows by (3.8), (3.9) and Theorem 1.6 that u_j is strongly convergent in $L^2(J,H)$ to v (say), and also that

$$u_j'(t) = \int_0^T \frac{\partial}{\partial t} G_A(t,\sigma) g_j(\sigma) d\sigma$$

converges strongly in $L^2(J,H)$ to w (say), as $j \to \infty$. (We note that the above argument follows the pattern of I Proposition 1.4.)

Because $u_j \to v$ strongly in $L^2(J,H)$ and $B_n(t) \to B(t)$ weakly in $\mathcal{B}(L^2(J,H))$, it follows just as in (2.20) that

$$g_j = f_j(t) - (B_j(t) - \ell I)u_j(t) \to f(t) - (B(t) - \ell I)v(t)$$

weakly in $L^2(J,H)$.

The compactness of the integral operators in Theorem 1.6 and (3.9) implies firstly that

$$u_j \to \int_0^T G_A(t,\sigma)[f(\sigma) - (B(\sigma) - \ell I)v(\sigma)]d\sigma$$

strongly, so that

$$v(t) = \int_0^T G_A(t,\sigma)[f(\sigma) - (B(\sigma) - \ell I)v(\sigma)]d\sigma \,,$$

hence by (2.6) and Theorem 2.2, $v(t)$ is the unique weak solution of the boundary value problem (2.1), (1.2), i.e. $v(t) = u(t)$. Also

$$u_j'(t) \to \int_0^T \frac{\partial}{\partial t} G_A(t,\sigma)[f(\sigma) - (B(\sigma) - \ell I)u(\sigma)]d\sigma = u'(t)$$

strongly in $L^2(J,H)$, (see (2.9)), so that $w(t) = u'(t)$. In conclusion we have that $u_j \to u$ in $\mathcal{H}$.

Since every sequence of values of n has a subsequence j for which $u_j \to u$ in $\mathcal{H}$, it follows that the whole sequence converges to u. This completes the proof $\square$.

In view of equation (3.1) we now state the assumptions on $B(t,x,y)$ and $f(t,x,y)$.

(B) For each $(x,y) \in H \times H$ and almost every $t \in J$, $B(t,x,y) \in \mathcal{B}(H)$. We assume that $B(t,x,y)$ is strongly measurable in t, for each $(x,y) \in H \times H$ and strongly continuous in (x,y), for almost every $t \in J$. It is supposed further, that $B(t,\eta(t),\rho(t))$ satisfies (AIII.3) for every $(\eta(t),\rho(t)) \in L^2(J,H) \times L^2(J,H)$ and that for every $\psi(t) \in \mathcal{H}^{1,2}$ of (3.6), $B(t,\psi(t),\psi'(t))$ belongs to the set $\mathcal{S}$ of (2.17).

(F) (α) Assume that $f(t,x,y)$ is measurable in $t \in J$ for each $(x,y) \in H \times H$ and continuous in (x,y) for almost every $t \in J$. (β) For each $(\eta(t),\rho(t)) \in L^2(J,H) \times L^2(J,H)$ suppose that $f(t,\eta(t),\rho(t))$ belongs to $L^2(J,H)$. (γ) Lastly we assume that for every sequence (x_n) in $\mathcal{H}^{1,2}$ with $\|x_n\| \le n$ one has

$$\liminf_{n \to \infty} n^{-1} \|f(t,x_n,x_n')\|_{L^2(J,H)} = 0 \quad . \tag{3.10}$$

<u>Remark.</u> 1) The asymptotic sublinearity condition in (3.10) is

slightly stronger than the corresponding ones in the Theorems I 3.1 and II 2.1, because the operator in (2.13) is uniformly bounded in $L^2(J,H)$ only, and not in H .

2) The operators $\tilde{B}(t,\psi(t),\psi'(t))$ are uniformly bounded for $\psi \in \mathcal{H}$.

3) The proof of I Proposition 1.3 can easily be extended to the more general situation described in (F) $(\alpha),(\beta)$. Hence the conclusions of I Proposition 1.3 also hold in this situation, i.e. the operator which assigns to $(\eta(t),\rho(t)) \in L^2(J,H) \times L^2(J,H)$ the element $f(t,\eta(t),\rho(t))$ in $L^2(J,H)$ is continuous and transforms bounded sets into bounded sets, c.f. Martin [51; p164].

Here is then the main result, namely

Theorem 3.2. *Assume that* (AIII.1) - (AIII.4), (B) *and* (F) *are satisfied and that* $\ell \geq 0$ *is sufficiently large,* (*see* Theorem 1.4). *Further suppose that for every* $B(t)$ *in* $\mathcal{S}$ *of* (2.17) *the homogeneous equation in* (2.3) *together with the boundary conditions* (1.2) *admit only the weak solution* $u(t) \equiv 0$.

Then eqn. (3.1) *has a weak solution* $u(t)$ *which satisfies the boundary conditions* (1.2). *The solution* $u(t)$ *satisfies the integral equation*

$$u(t) = \int_0^T G_A(t,\sigma)[f(\sigma,u,u') - (B(\sigma,u,u') - \ell I)u(\sigma)]d\sigma . \tag{3.11}$$

Proof. Let $\psi(t) \in \mathcal{H}$. According to the assumptions the equation

$$x'' = (A + B(t,\psi,\psi'))x - f(t,\psi,\psi') \tag{3.12}$$

together with the boundary conditions (1.2) has a unique weak solution $u(t)$, by Theorem 2.2, and it satisfies

$$u(t) = \int_0^T G_A(t,\sigma)[f(\sigma,\psi,\psi') - (B(\sigma,\psi,\psi') - \ell I)u(\sigma)]d\sigma . \tag{3.13}$$

Thus to each $\psi(t) \in H$ one can assign the unique weak solution $u(t)$ of the boundary value problem (3.12), (1.2) . The solution $u(t)$ belongs to H since $u'(t) \in C(J,H)$. For this we refer to part (β) in the proof of Theorem 2.1 and to (α) following (3.7) . This defines a map

$$W : H \to H \ , \ W(\psi) = u \ .$$

In order to apply Schauder's fixed point theorem, we will verify that (a) W is compact, (b) W is continuous and (c) there exists a ball K_n in H such that $W(K_n) \subset K_n$. Throughout the proof we will abbreviate $f(t,\psi_n,\psi_n')$ to $f_n(t)$ and $B(t,\psi_n,\psi_n')$ to $B_n(t)$, for $\psi_n \in H$.

(a) W *is compact.* Let (ψ_n) be a bounded sequence in H . By I Proposition 1.3, $(f_n(t))$ is bounded in $L^2(J,H)$ (by R say). By (2.11) one has

$$u_n(t) = \int_0^T G_n(t,\sigma)f_n(\sigma)d\sigma \ . \tag{3.14}$$

Since $B_n(t)$ belong to S , Theorem 2.3 implies that

$$\|u_n\|_{L^2(J,H)} \le N\|f_n\| \le NR \ ,$$

i.e. (u_n) is a bounded sequence in $L^2(J,H)$. It follows that

$$g_n(t) = f_n(t) - (B_n(t) - \ell I)u_n(t)$$

is a bounded sequence in $L^2(J,H)$. Let (g_j) denote a weakly convergent subsequence thereof. The compactness of the integral operators in Theorem 1.6 and (3.9) implies that both

$$u_j(t) = \int_0^T G_A(t,\sigma)g_j(\sigma)d\sigma$$

and

$$u_j{}'(t) = \int_0^T \frac{\partial}{\partial t} G_A(t,\sigma) g_j(\sigma) d\sigma$$

are convergent in $L^2(J,H)$. (The argument follows again the pattern of I Proposition 1.4.) Since H is a Hilbert space it follows that the subsequence $(W(\psi_j))$ is convergent in H , i.e. W is compact.

(b) W *is continuous.* Let $\psi_n \to \psi$ in H . I Proposition 1.3 implies that $f_n(t)$ converges to $f(t,\psi,\psi')$ in $L^2(J,H)$. For any $z(t) \in L^2(J,H)$, $B(t,x,y)z(t)$ satisfies the assumptions (α), (β) of (F) and hence I Proposition 1.3 applies to $B(t,x,y)z(t)$. Therefore as $n \to \infty$

$$B_n(t)z(t) \to B(t,\psi,\psi')z(t)$$

strongly in $L^2(J,H)$. Thus $B_n(t)$ converges strongly in $\mathcal{B}(L^2(J,H))$ to $B(t,\psi,\psi')$. Theorem 3.1 now implies that $u_n(t)$ converges to $u(t)$ strongly in H , i.e. W is continuous.

(c) *Letting* $K_n = \{x \in H \mid \|x\| \le n\}$, *there exists a positive integer* n , *such that* $W(K_n) \subset K_n$.

Suppose this is not so. Hence for every integer n , we can find $x_n \in K_n$ such that $\|W(x_n)\| > n$. We will estimate the norm of $W(x_n)$ and then use (3.10) to obtain a contradiction. Thus

$$\|W(x_n)\|_H^2 = \int_0^T \|u_n(t)\|_H^2 + \|u_n{}'(t)\|_H^2 \, dt ,$$

and by (3.13), (3.14) we have

$$u_n(t) = \int_0^T G_A(t,\sigma) g_n(\sigma) d\sigma$$

and by (2.9)

$$u_n'(t) = \int_0^T \frac{\partial}{\partial t} G_A(t,\sigma) g_n(\sigma) d\sigma$$

where we have set

$$g_n(\sigma) = f_n(\sigma) - (B_n(\sigma) - \ell I) \int_0^T G_n(\sigma,\lambda) f_n(\lambda) d\lambda .$$

The operator functions $G_A(t,\sigma)$ and $(\partial/\partial t)G_A(t,\sigma)$, $\sigma \neq t$, are uniformly bounded in t and σ (by C say) , using (1.42), (1.39) and (2.8). Thus

$$\begin{aligned}
\|W(x_n)\|_H^2 &\leq \int_0^T \{ (\int_0^T \|G_A(t,\sigma) g_n(\sigma)\| d\sigma)^2 + (\int_0^T \|\frac{\partial}{\partial t} G_A(t,\sigma) g_n(\sigma)\| d\sigma)^2 \} dt \\
&\leq 2TC^2 (\int_0^T \|g_n(\sigma)\| d\sigma)^2 \\
&\leq 4TC^2 \{ (\int_0^T \|f_n(\sigma)\| d\sigma)^2 + (\int_0^T \|B_n(\sigma) - \ell I\| \| \int_0^T G_n(\sigma,\lambda) f_n(\lambda) d\lambda \| d\sigma)^2 \} \\
&\leq 4TC^2 \{ T\|f_n\|^2_{L^2(J,H)} + P^2 (\int_0^T \| \int_0^T G_n(\sigma,\lambda) f_n(\lambda) d\lambda \| d\sigma)^2 \} \\
&\leq 4T^2C^2 \{ \|f_n\|^2_{L^2(J,H)} + P^2 \| \int_0^T G_n(\sigma,\lambda) f_n(\lambda) d\lambda \|^2_{L^2(J,H)} \} \\
&\leq 4T^2C^2 (1 + P^2Q^2) \|f_n\|^2_{L^2(J,H)} .
\end{aligned}$$

The last step is based on Theorem 2.3. In our notation, $f_n(t) = f(t, x_n, x_n')$, thus for all n

$$1 < \|W(x_n)\|_H^2 \, n^{-2} \leq 4T^2C^2 (1+P^2Q^2) (n^{-1} \|f(t,x_n,x_n')\|_{L^2(J,H)})^2 .$$

This contradicts (3.10).

We note that the above proof is slightly different from the corresponding one in I Theorem 3.1 and refer to Remark (1) which precedes the statement of this theorem.

It follows by Schauder's theorem that W has a fixed point $y(t)$. It satisfies (3.13) and, because $Wy = y$, also (3.11) . Using the proof of Theorem 2.1, see (2.6), we can conclude that $y(t)$ is a weak solution of the boundary value problem (3.1), (1.2) . This completes the proof □ .

In the subsequent section we will illustrate the result just obtained.

4. EXAMPLES

In this section we will illustrate the results obtained so far. It will consist of three parts. Firstly an operator A will be described representing an elliptic boundary value problem and satisfying (AIII.1) and (AIII.2). Conditions will then be given such that the uniqueness assumption of Theorem 3.2 holds. The type of condition which we will use first (in Theorem 4.1), is derived from Becker [6] and is based on Landesman and Lazer [44]. A different condition is then considered, based on the accretiveness of the operators $A + B(t)$, which implies uniqueness of solution of equation (2.1) in the case of the first boundary value problem (Theorem 4.2).

Secondly an integral operator will be characterized which satisfies the conditions imposed on the perturbation $B(t, x,y)$ in Theorem 3.2.

Lastly we will present a non-differential operator A pertaining to the conditions (AIII.1) and (AIII.2).

4.1. *Elliptic case.* The operator A representing an elliptic boundary value problem is taken from Tanabe [72; p77-87]. Denote by $H_m(\Omega)$, the set of all functions which, together with their derivatives to order m in the sense of distribution, belong to the Lebesgue space $H = L^2(\Omega)$, where Ω is a bounded region of class C^m in $\mathbb{R}^n$. We denote the norm of f in $H_m(\Omega)$ by $\|f\|_m$, c.f. [72; p11-14].

Let the linear, differential operator defined in Ω by

$$T(x,D) = \Sigma_{|\alpha|\leq 2p}\, a_\alpha(x) D^\alpha$$

be properly elliptic, and assume that its coefficients belong to $L^{\infty}(\Omega)$. In particular the coefficients $a_{\alpha}(x)$ with $|\alpha| = 2p$ are assumed to be continuous in $\bar{\Omega}$.

Next let

$$B_j(x,D) = \Sigma_{|\beta|\le m_j} b_{j\beta}(x)D^{\beta}$$

for each $j = 1,\dots,p$ be a linear differential operator of order $m_j < 2p$, whose coefficients are defined in $\partial\Omega$ and belong to $C^{2p-m_j}(\partial\Omega)$. Assume further that these boundary differential operators are normal, (see [72; p77-79]).

The operator T is defined as follows

$$\mathcal{D}(T) = \{u \in H_{2p}(\Omega) \,|\, B_j(x,D)u(x) = 0 \text{ on } \partial\Omega \,,\; j = 1,\dots p\} \tag{4.1}$$

and

$$(Tu)(x) = T(x,D)u(x) \quad \text{for} \quad u \in \mathcal{D}(T) .$$

We also assume that the operator $-T$ satisfies the two conditions stated in Theorem 3.8.1 in [72;p82] for the angle $\theta = 0$, so that the half-line $\arg(\lambda) = 0$ is a ray of minimal growth of the resolvent of $-T$.

Thus the operator $-T$ is closed and densely defined in $L^2(\Omega)$ and the estimate $\|(-T-\lambda I)^{-1}\| \le k'/\lambda$ holds for $\lambda > k'$ and k' some positive number, referring to [72; p82].

Choose $w > 0$ so that $k = k' + w \ge 1$ and set $\lambda = k + (\lambda - k) = k + \mu$. It follows that

$$\|((T+kI) + \mu I)^{-1}\| \le k'/k+\mu \le k'/1+\mu \;,\; \mu \ge 0 .$$

Therefore the operator

$$A = T + kI \tag{4.2}$$

satisfies (AIII.1). Concerning (AIII.2) we apply Rellich's Lemma, [72; p14]. Since 0 is in the resolvent set of A we can conclude, just as in Chapter I, that A^{-1} is compact.

Referring to Theorem 1.5, we choose $\ell \geq 0$ sufficiently large to ensure that the regular boundary value problem (1.1), (1.2) has a unique weak solution for $f \in L^2(J,L^2(\Omega)) = L^2(J\times\Omega)$, (see Balakrishnan [3; p134]), with A given by (4.1), (4.2) .

The operator in $L^2(J\times\Omega)$

$$\int_0^T G_\ell(t,\sigma) \text{ - } d\sigma \tag{4.3}$$

is bounded by $TC/\sqrt{\ell+1}$, by virtue of (1.46). In accordance with (2.17) we now set

$$S_\ell = \{B(t)\in M(J,L^2(\Omega)) \,|\, \|B(t)-\ell I\| \leq \alpha < \sqrt{\ell+1}/TC \ , \quad \text{for a.e. } t \in J\} \ , \tag{4.4}$$

for $B(t)$ satisfying (AIII.3) .

It follows that whenever $B(t)$ belongs to S_ℓ and $\ell \geq 0$ is sufficiently large, the homogeneous equation

$$x'' = (A+B(t))x = (A+\ell I)x + (B(t)-\ell I)x$$

has only the weak solution $u(t) \equiv 0$ which satisfies the regular boundary conditions (1.2). For, suppose $u(t)$ were such a solution. By virtue of Theorem 1.5, it would satisfy

$$u(t) = -\int_0^T G_\ell(t,\sigma)(B(\sigma)-\ell I)u(\sigma)d\sigma \ ,$$

so that for the norms in $L^2(J\times\Omega)$ we would obtain

$$\|u\| \leq TC(\ell+1)^{-1/2}\|(B(t)-\ell I)u(t)\| < \|u\| \ .$$

This is a contradiction, so $u(t) \equiv 0$ as required. We have therefore proved

Theorem 4.1. *Let the operator* A *be given by* (4.1), (4.2) *and suppose that* $f(t,u,v)$ *satisfies condition* (F) *and* $B(t,u,v)$ *satisfies condition* (B), *with* S *replaced by* S_ℓ *given by* (4.4), *where* ℓ *is chosen in accordance with* Theorem 1.4. *The equation*

$$\frac{\partial^2 u}{\partial t^2} = [T(x,D) + B(t,u,\partial u/\partial t) + kI]u(t,x) - f(t,x;u,\partial u/\partial t)$$

then has a weak solution $u(t,x)$ *which satisfies the regular boundary conditions*

$$\alpha_{i1}u(0,x) + \alpha_{i2}\partial u/\partial t(0,x) + \beta_{i1}u(T,x) + \beta_{i2}\partial u/\partial t(T,x) = 0$$

for almost every $x \in \Omega$ *and* $i = 1,2$. *The weak solution also satisfies equation* (3.11).

Remark. 1) Since $u(t,x)$ is a weak solution, $u(t,x)$ need not belong to the domain of A, for any $t \in J$.
2) By virtue of (4.4) the perturbation may vary in an ever increasing neighbourhood of ℓI, provided the value of $\ell \geq 0$ is increased.

For the case of the first boundary value problem, i.e. $x(0) = 0 = x(T)$, we give a condition which guarantees uniqueness of solution of equation (2.1).

Theorem 4.2. *Suppose that* H *is a real Hilbert space and that the operator* A *in* H *satisfies* (AIII.1). *We assume that* $B(t) \in M(J,H)$

is strongly Hölder continuous and that $A + B(t)$ *is accretive for every* $t \in (0,T)$ *. The equation*

$$x'' = (A+B(t))x$$

with
$$x(0) = 0 = x(T)$$

then has only the zero solution.

Proof. Suppose to the contrary that $v(t) \not\equiv 0$ is a solution of the above equation with $v(0) = 0 = v(T)$. Hence $(\ell I-B(t))v(t)$ is a given function in $L^2(J,H)$. By virtue of Theorem 1.5, $v(t)$ satisfies eqn. (2.5). From (2.9) we see that $v(t) \in C^1(J,H)$. Our first assumption on $B(t)$ implies, see Krein [43; p267], that $v(t)$ has a second derivative on $(0,T)$, that its values lie in $\mathcal{D}(A)$ for $t \in (0,T)$ and that $v(t)$ satisfies eqn. (2.3) in the strict sense in the interval $(0,T)$.

Set $h(t) = 2^{-1}\|v(t)\|^2 = 2^{-1}(v(t),v(t))$. Then $h''(t) = (v'',v) + (v',v')$, and $(v'',v) = ((A+B(t))v,v) \geq 0$ for all $t \in (0,T)$. Hence $h'' \geq \|v'\|^2$ which means that $h(t)$ is convex on J. It follows that $\|v(t)\|^2 \leq \max\{\|v(0)\|^2, \|v(T)\|^2\} = 0$ □ .

The above convexity argument is well known and was applied for example by Bruck [9; p161].

4.2 *An integral operator representing the perturbation* $B(t,u,v)$. An integral operator is described which satisfies the conditions imposed on $B(t,u,v)$ in Theorem 3.2. For simplicity we will assume that $\ell = 0$ in the following. We use the same notation as in Section 4.1.

We assume that $h : J \times \Omega \times \Omega \to \mathbb{R}$ is a real valued function which is measurable in t , for every $(x,y) \in \Omega \times \Omega$ and continuous in $\bar{\Omega} \times \bar{\Omega}$, for every $t \in J$. Let K denote the weighted $L^2(\Omega)$ functions, where we have taken the square of $h(t,x,y)$ to be the weight function (viewed as a function of y) .

Theorem 4.3. *Assume that for every* $t \in J$ *and* $x \in \Omega$, *the map*

$$p : L^2(\Omega) \times L^2(\Omega) \to K$$

is continuous and that the following estimates hold. For every $t \in J$ *and any* $u,v \in L^2(\Omega)$

$$\int_\Omega |p(u,v)(y)h(t,x,y)|^2 dy \le \ell(x) \ , \tag{4.5}$$

where $\ell(x)$ *is integrable over* Ω , *and*

$$\int_\Omega\int_\Omega |p(u,v)(y)h(t,x,y)|^2 dydx \le \alpha^2 \ . \tag{4.6}$$

The integral operator in $L^2(\Omega)$ *defined by*

$$B(t,u,v) = \int_\Omega p(u,v)(y)h(t,x,y) \cdot dy \tag{4.7}$$

then satisfies the conditions in Theorem 3.2.

Proof. For any $u,v \in L^2(\Omega)$ and $t \in J$ the operator is bounded by α by virtue of (4.6), using Hölder's inequality. From this it follows immediately that for any $\psi(t,x)$, $\rho(t,x) \in L^2(J\times\Omega)$

$$\| B(t,\psi,\rho)\| \le \alpha$$

for almost every $t \in J$. The measurability follows from the assumption on $h(t,x,y)$. The continuity of p and (4.5) allow us to apply the dominated convergence theorem and so to prove the continuity of $B(t,u,v)$ in u and v .

In view of Theorem 4.1, (see (4.4)), $B(t,u,v)$ given by (4.7) fulfils the conditions of Theorem 3.2, if we let $\alpha > 0$ be sufficiently small in (4.6). This completes the proof □ .

4.3 *A non-differential operator* A . Let (λ_n) be a sequence of complex numbers satisfying

$$\inf_n \mathrm{Re}(\lambda_n) = d > 0 \quad \text{and} \quad \sum_n |\lambda_n|^{-2} < \infty$$

and let (ϕ_n) denote a complete orthonormal sequence in a separable Hilbert space H , c.f. Curtain and Pritchard [16; p21].

Theorem 4.4. *The operator* $-A$ *defined by*

$$\mathcal{D}(A) = \{z \in H \mid \sum_n |\lambda_n(z,\phi_n)|^2 < \infty\}$$

$$Az = \sum_n \lambda_n(z,\phi_n)\phi_n \ , \quad \textit{for} \quad z \in \mathcal{D}(A)$$

satisfies (AIII.1) *and* (AIII.2).

Proof. In verifying (AIII.1), we follow [16; p21].

(i) $-A$ *satisfies* (AIII.1). All sequences (z_p) with $(z_p,\phi_n) = 0$ for n sufficiently large lie in $\mathcal{D}(A)$ and form a dense set in H .

Now let (z_p) be a sequence in $\mathcal{D}(A)$ with $z_p \to z_0$ and $Az_p \to y_0$ as $p \to \infty$. Since the sequence (Az_p) is bounded we have for $p = 1,2,\ldots$

$$\sum_n |\lambda_n(z_p,\phi_n)|^2 \leq M \ .$$

Since $\ell^2(\mathbb{C})$ is a Hilbert space it therefore follows that

$$\sum_n |\lambda_n(z_0,\phi_n)|^2 \leq M$$

and so $Az_0 = y_0$, showing that A is closed.

For $\lambda \le 0$ and $z \in \mathcal{D}(A)$ consider

$$(\lambda I-A)z = y = \sum_n (y,\phi_n)\phi_n .$$

It follows that

$$(\lambda-\lambda_n)(z,\phi_n) = (y,\phi_n)$$

and thus

$$z = (\lambda I-A)^{-1}y = \sum_n (\lambda-\lambda_n)^{-1}(y,\phi_n)\phi_n .$$

In the following we will need the estimates

$d \le \mathrm{Re}(\lambda_n) \le |\lambda_n|$ and $|\lambda| + d \le |\lambda-\lambda_n|$ for $\lambda \le 0$ and all n .

Therefore

$$\|(\lambda I-A)^{-1}y\|^2 = \sum|\lambda-\lambda_n|^{-2}|(y,\phi_n)|^2 \le$$

$$\le (d+|\lambda|)^{-2}\|y\|^2 \le C^2(1+|\lambda|)^{-2}\|y\|^2 ,$$

and hence (AIII.1) is verified.

(ii) A^{-1} *is compact.* Let (x_j) be a sequence weakly converging to y in H . Thus for each n we have $(x_j-y,\phi_n) \to 0$ as $j \to \infty$, and since (x_j) is a bounded sequence $|(x_j-y,\phi_n)| \le \|x_j-y\| \le C$.

We will show that

$$A^{-1}x_j - A^{-1}y = \sum_n \lambda_n^{-1}(x_j-y,\phi_n)\phi_n \to 0$$

as $j \to \infty$. Thus

$$\|A^{-1}x_j - A^{-1}y\|^2 = \sum_n |\lambda_n|^{-2}|(x_j-y,\phi_n)|^2$$

$$\le C^2\sum|\lambda_n|^{-2} < \infty .$$

To a given $\varepsilon > 0$ we can therefore find a positive integer N such that

$$\sum_{n=N+1}^{\infty} |\lambda_n|^{-2} |(x_j - y, \phi_n)|^2 < \varepsilon ,$$

independently of j. Further for every $n \leq N$, there exists j_n such that $j > j_n$ implies $|(x_j - y, \phi_n)|^2 < \varepsilon d^2 N^{-1}$. Let $k = \max\{j_n | n \leq N\}$, so that for $j > k$ one obtains

$$\|A^{-1}x_j - A^{-1}y\|^2 = \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} \right) |\lambda_n|^{-2} |(x_j - y, \phi_n)|^2$$

$$< \sum_{n=1}^{N} d^{-2} |(x_j - y, \phi_n)|^2 + \varepsilon < 2\varepsilon .$$

Hence $A^{-1}x_j$ converges strongly to $A^{-1}y$, showing that A^{-1} is compact $\square$.

APPENDIX

We now prove the assertions made in (1.7) - (1.11).

Concerning (1.7), by virtue of Balakrishnan [4;(5.3)p430], we have

$$V_\ell(t) = \frac{2}{\pi t^2}\int_0^\infty (A_\ell+\lambda I)^{-2}\{\sin\sqrt{\lambda}t - \sqrt{\lambda}t\cos\sqrt{\lambda}t\}d\lambda \ , \ t > 0 \tag{4.8}$$

$$V_\ell(0) = I \ .$$

The resolvent $(A_\ell+\lambda I)^{-1}$ is bounded by M/λ uniformly for $\ell \geq 0$, making use of (AIII.1). Therefore for any $\ell \geq 0$, $t > 0$ and a fixed $\lambda = \lambda_0 > 0$, we consider

$$\|V_\ell(t)\| \leq 2M^2\pi^{-1}\{\int_0^{\lambda_0 t^{-2}} + \int_{\lambda_0 t^{-2}}^\infty |\sin\sqrt{\lambda}t - \sqrt{\lambda}t\cos\sqrt{\lambda}t|(\lambda t)^{-2}d\lambda$$

$$\equiv 2M^2\pi^{-1}(I+II) \ .$$

This expression is of course uniformly bounded for $\ell \geq 0$. We will show that it is also uniformly bounded in $t \in [0,T]$. In the first integral we use a Taylor expansion for the trigonometric functions and obtain

$$I \leq \int_0^{\lambda_0 t^{-2}} (\lambda t)^{-2}\{\lambda^{3/2}\cdot t^3/3 + 6\lambda^{5/2}\cdot t^5/5!\}d\lambda = 2\lambda_0^{1/2}/3 + 4\lambda_0^{3/2}/5! \ .$$

For the second integral one has

$$II \leq \int_{\lambda_0 t^{-2}}^\infty (\lambda t)^{-2}(1+\sqrt{\lambda}t)d\lambda = \lambda_0^{-1} + 2\lambda_0^{-\frac{1}{2}} \ .$$

Thus for any $t > 0$ and $\ell \geq 0$, $V_\ell(t)$ is bounded by a constant which depends only on λ_0 and on M. This proves (1.7) □.

Concerning (1.8), we use the fact that $(A_\ell+\lambda I)^{-1} = (A+(\ell+\lambda)I)^{-1}$ is bounded by $M(1+\ell+\lambda)^{-1}$. By virtue of (4.8) we obtain for $t > 0$

$$\|V_\ell(t)\| \leq \frac{2M^2}{\pi t^2}\int_0^\infty (1+\sqrt{\lambda}t)(1+\ell+\lambda)^{-2}d\lambda = 2M^2\{(\pi t^2(\ell+1))^{-1} + (2t\sqrt{\ell+1})^{-1}\},$$

by a simple integration. Thus the result follows □.

Concerning (1.9), we have, by virtue of Krein [43;(5.8)p112], $\ell \geq 0$,

$$A_\ell^{-\frac{1}{2}} = \pi^{-1}\int_0^\infty \lambda^{-\frac{1}{2}}(A+(\ell+\lambda)I)^{-1}d\lambda,$$

and using (AIII.1) we obtain

$$\|A_\ell^{-\frac{1}{2}}\| \leq M\pi^{-1}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\ell+\lambda)^{-1}d\lambda = M/\sqrt{1+\ell},$$

which proves the statement □.

Concerning (1.10), (AIII.1) guarantees that both A^* and $A^{\frac{1}{2}*}$ are closed and densely defined. From Balakrishnan [3; p85] it follows that

$$(A^*)^{-1} = (A^{-1})^* \quad \text{and} \quad (A^{\frac{1}{2}*})^{-1} = (A^{-\frac{1}{2}})^*. \tag{4.9}$$

Let $y \in \mathcal{D}(A^*)$, $A^*y = z$. By (4.9) and since $A^{-\frac{1}{2}}$ is bounded, we have $y = (A^{-1})^*z = (A^{-\frac{1}{2}}\cdot A^{-\frac{1}{2}})^*z = (A^{-\frac{1}{2}})^*(A^{-\frac{1}{2}})^*z = (A^{\frac{1}{2}*})^{-1}(A^{\frac{1}{2}*})^{-1}z$. Therefore $y \in \mathcal{D}(A^{\frac{1}{2}*})$ and $A^{\frac{1}{2}*}y \in \mathcal{D}(A^{\frac{1}{2}*})$, and $A^{\frac{1}{2}*}\cdot A^{\frac{1}{2}*}y = A^*y$.

Now let $z \in \mathcal{D}(A^{\frac{1}{2}*}\cdot A^{\frac{1}{2}*})$ and $v = A^{\frac{1}{2}*}\cdot A^{\frac{1}{2}*}z$ so that $z = (A^{-\frac{1}{2}})^*(A^{-\frac{1}{2}})^*v = (A^{-\frac{1}{2}}\cdot A^{-\frac{1}{2}})^*v = (A^{-1})^*v = (A^*)^{-1}v$. Therefore $z \in \mathcal{D}(A^*)$ and $A^*z = A^{\frac{1}{2}*}\cdot A^{\frac{1}{2}*}z$ and we have verified (1.10) □.

Concerning (1.11), let $z \in \mathcal{D}(A^*)$. Then by what we have shown above

$A^{\frac{1}{2}*}z = y \in \mathcal{D}(A^{\frac{1}{2}*})$ and $z = (A^{\frac{1}{2}*})^{-1}y$. Now let $y \in \mathcal{D}(A^{\frac{1}{2}*})$, $A^{\frac{1}{2}*}y = w$ and $z = (A^{\frac{1}{2}*})^{-1}y$. Then $(A^*)^{-1}w \in \mathcal{D}(A^*)$ and by (4.9), $(A^*)^{-1}w = (A^{-\frac{1}{2}})^*(A^{-\frac{1}{2}})^*w = (A^{\frac{1}{2}*})^{-1}y = z$. Thus (1.11) is verified. With this the appendix is complete $\square$.

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